

THE MANGA GUIDE TO

COMICS  
INSIDE!

# CALCULUS

HIROYUKI KOJIMA  
SHIN TOGAMI  
BECOM CO., LTD.





## PRAISE FOR THE MANGA GUIDE SERIES

“Highly recommended.”

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“Stimulus for the next generation of scientists.”

—SCIENTIFIC COMPUTING

“A great fit of form and subject. Recommended.”

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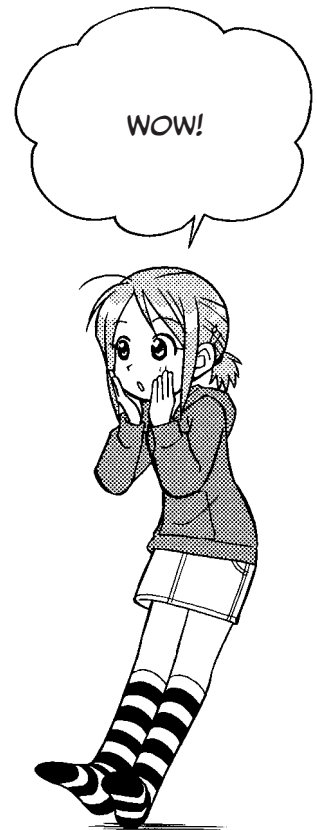
—THE SAN FRANCISCO EXAMINER

“A lot of fun to read. The interactions between the characters are lighthearted, and the whole setting has a sort of quirkiness about it that makes you keep reading just for the joy of it.”

—HACK A DAY

“*The Manga Guide to Databases* was the most enjoyable tech book I’ve ever read.”

—RIKKI KITE, LINUX PRO MAGAZINE





THE MANGA GUIDE™ TO CALCULUS



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**CALCULUS**

HIROYUKI KOJIMA  
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# PREFACE

There are some things that only manga can do.

You have just picked up and opened this book. You must be one of the following types of people.

The first type is someone who just loves manga and thinks, “Calculus illustrated with manga? Awesome!” If you are this type of person, you should immediately take this book to the cashier—you won’t regret it. This is a very enjoyable manga title. It’s no surprise—Shin Togami, a popular manga artist, drew the manga, and Becom Ltd., a real manga production company, wrote the scenario.

“But, manga that teaches about math has never been very enjoyable,” you may argue. That’s true. In fact, when an editor at Ohmsha asked me to write this book, I nearly turned down the opportunity. Many of the so-called “manga for education” books are quite disappointing. They may have lots of illustrations and large pictures, but they aren’t really manga. But after seeing a sample from Ohmsha (it was *The Manga Guide to Statistics*), I totally changed my mind. Unlike many such manga guides, the sample was enjoyable enough to actually read. The editor told me that my book would be like this, too—so I accepted his offer. In fact, I have often thought that I might be able to teach mathematics better by using manga, so I saw this as a good opportunity to put the idea into practice. I guarantee you that the bigger manga freak you are, the more you will enjoy this book. So, what are you waiting for? Take it up to the cashier and buy it already!

Now, the second type of person is someone who picked up this book thinking, “Although I am terrible at and/or allergic to calculus, manga may help me understand it.” If you are this type of person, then this is also the book for you. It is equipped with various rehabilitation methods for those who have been hurt by calculus in the past. Not only does it explain calculus using manga, but the way it explains calculus is fundamentally different from the method used in conventional textbooks. First, the book repeatedly

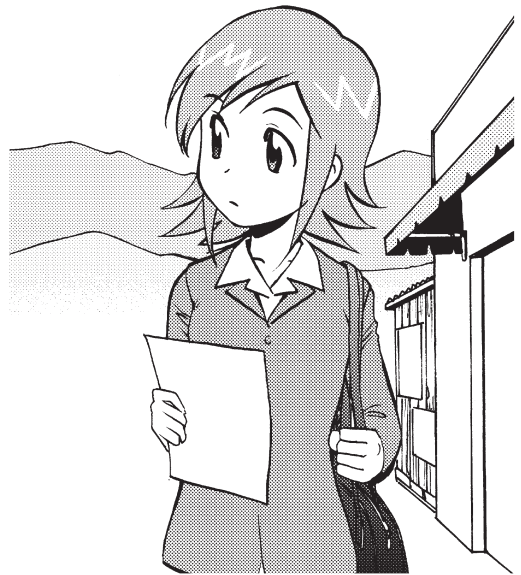
presents the notion of what calculus really does. You will never understand this through the teaching methods that stick to *limits* (or  $\varepsilon$ - $\delta$  logic). Unless you have a clear image of what calculus really does and why it is useful in the world, you will never really understand or use it freely. You will simply fall into a miserable state of memorizing formulas and rules. This book explains all the formulas based on the concept of the *first-order approximation*, helping you to visualize the meaning of formulas and understand them easily. Because of this unique teaching method, you can quickly and easily proceed from differentiation to integration. Furthermore, I have adopted an original method, which is not described in ordinary textbooks, of explaining the differentiation and integration of trigonometric and exponential functions—usually, this is all Greek to many people even after repeated explanations. This book also goes further in depth than existing manga books on calculus do, explaining even Taylor expansions and partial differentiation. Finally, I have invited three regular customers of calculus—physics, statistics, and economics—to be part of this book and presented many examples to show that calculus is truly practical. With all of these devices, you will come to view calculus not as a hardship, but as a useful tool.

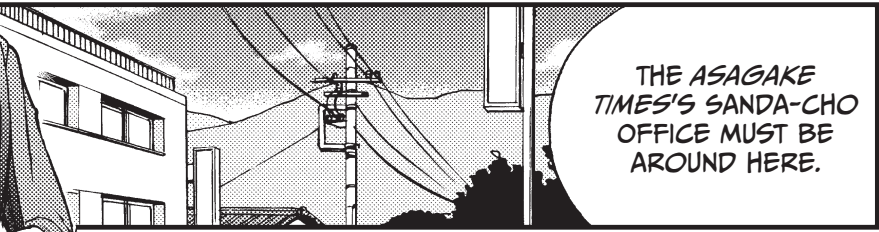
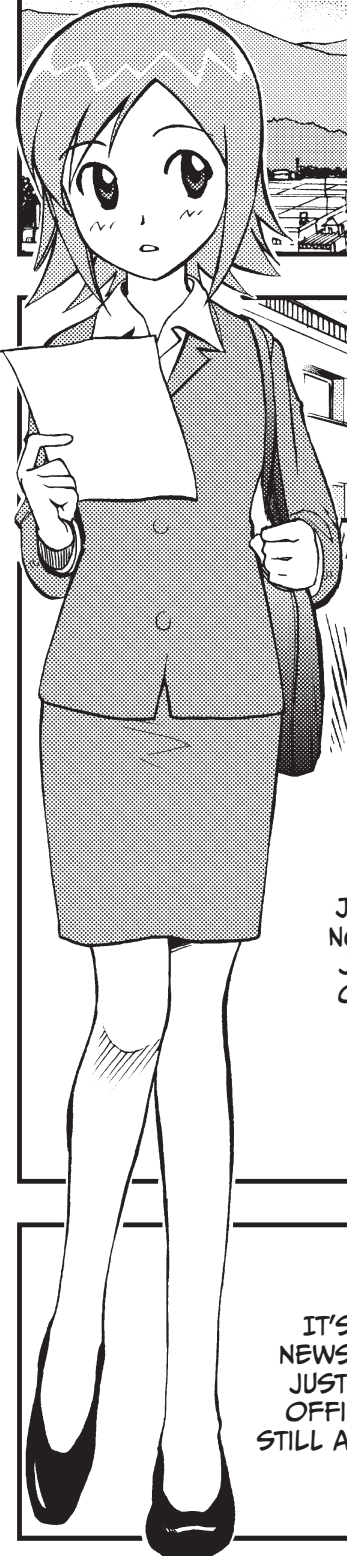
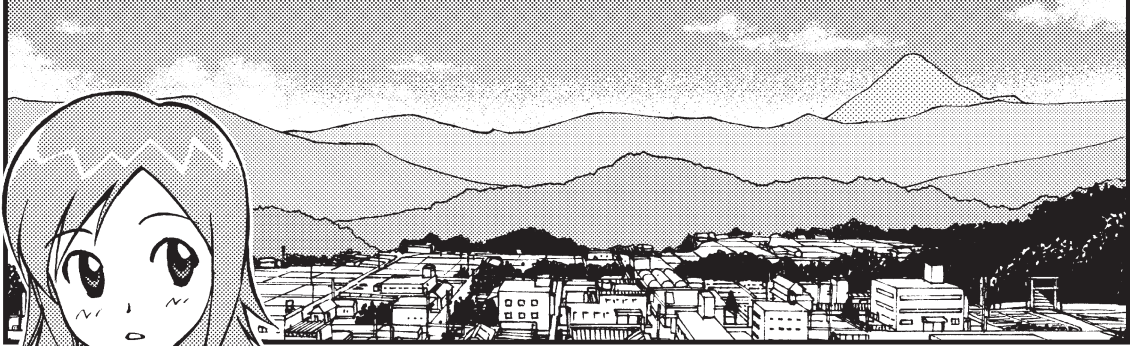
I would like to emphasize again: All of this has been made possible because of manga. Why can you gain more information by reading a manga book than by reading a novel? It is because manga is visual data presented as animation. Calculus is a branch of mathematics that describes dynamic phenomena—thus, calculus is a perfect concept to teach with manga. Now, turn the pages and enjoy a beautiful integration of manga and mathematics.

HIROYUKI KOJIMA  
NOVEMBER 2005

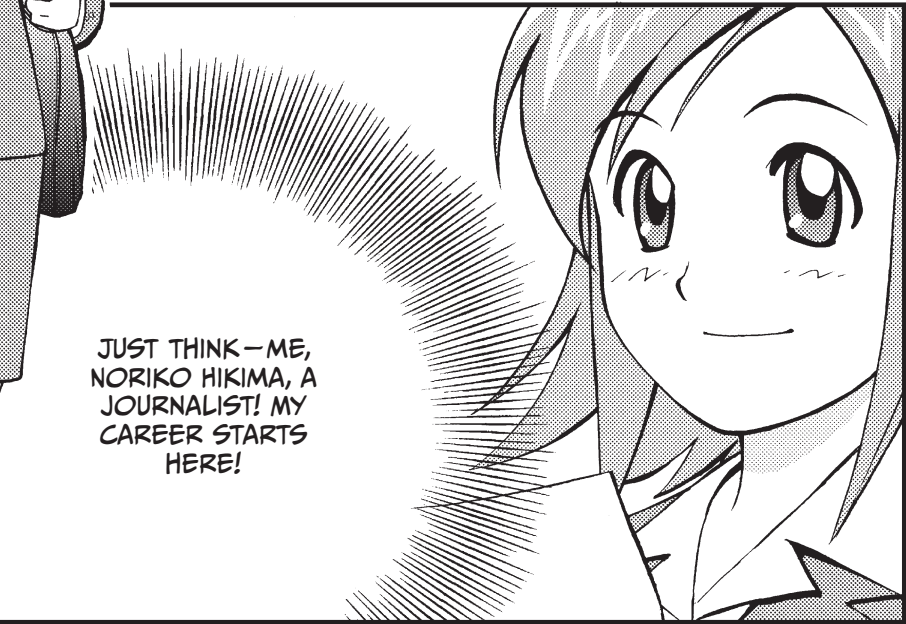
*NOTE: For ease of understanding, some figures are not drawn to scale.*

# PROLOGUE: WHAT IS A FUNCTION?





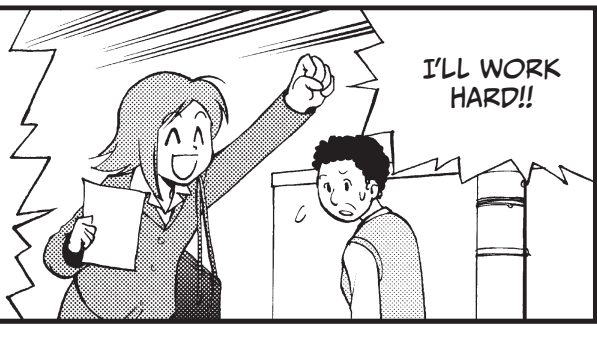
THE ASAGAKE TIMES'S SANDA-CHO OFFICE MUST BE AROUND HERE.



JUST THINK—ME, NORIKO HIKIMA, A JOURNALIST! MY CAREER STARTS HERE!

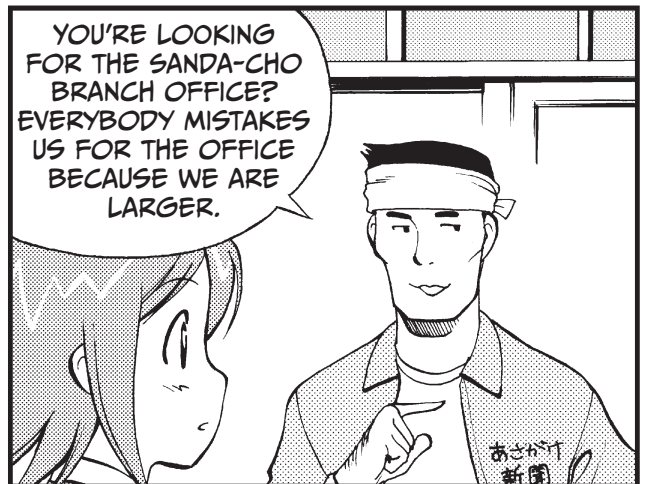
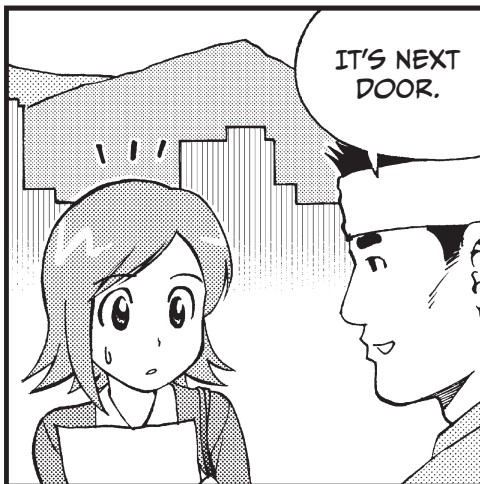
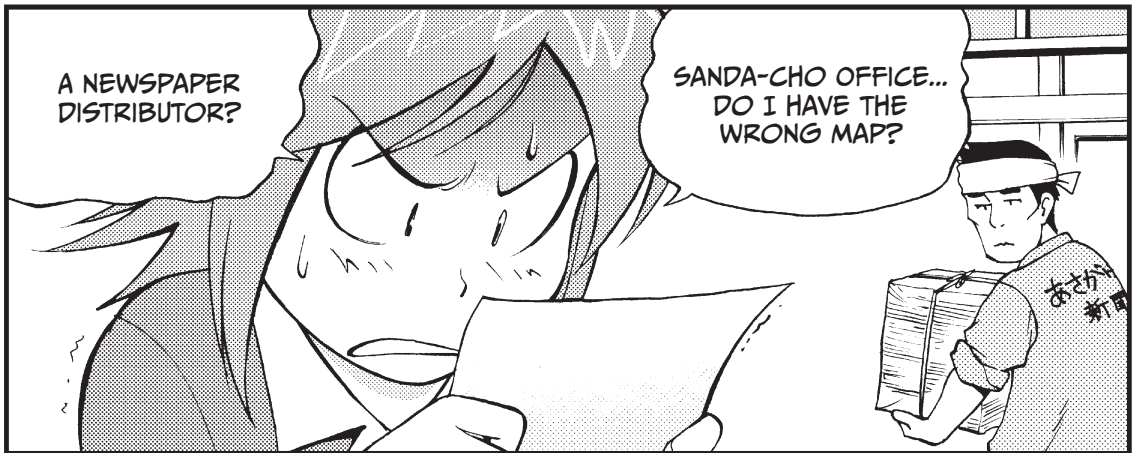


IT'S A SMALL NEWSPAPER AND JUST A BRANCH OFFICE. BUT I'M STILL A JOURNALIST!

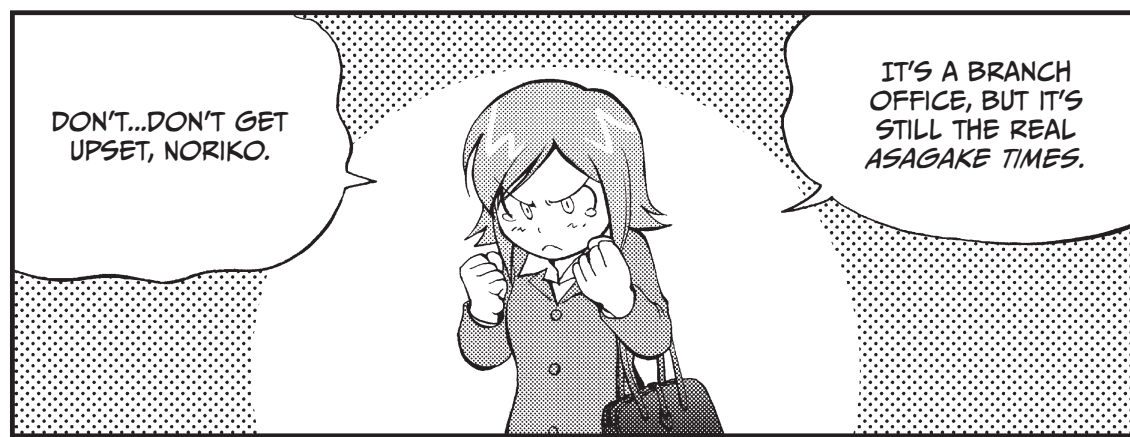
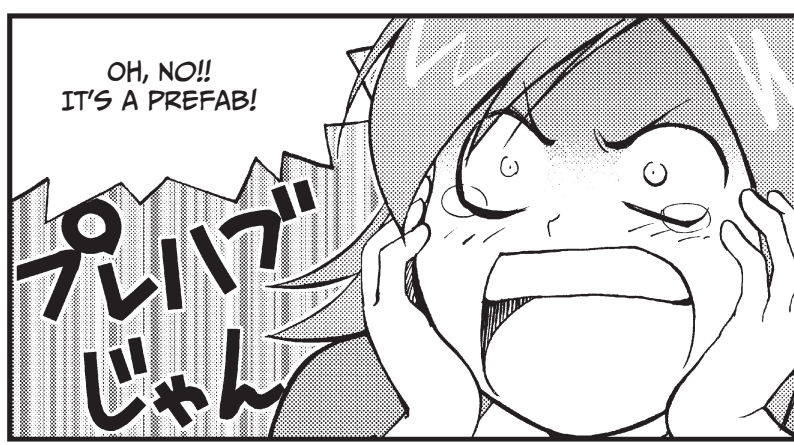
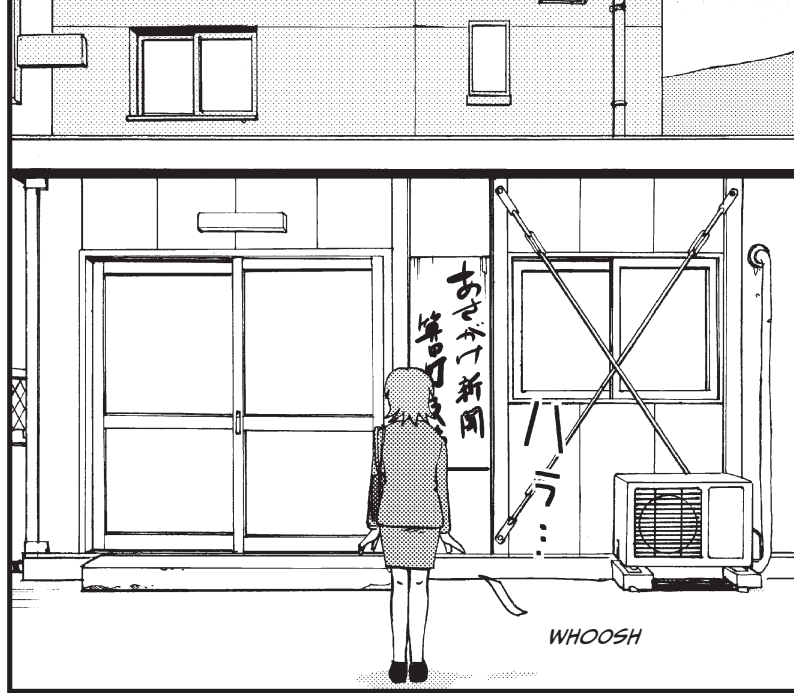


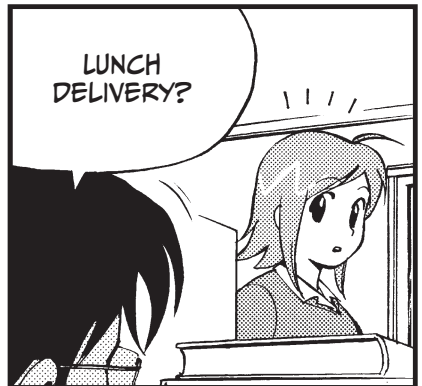
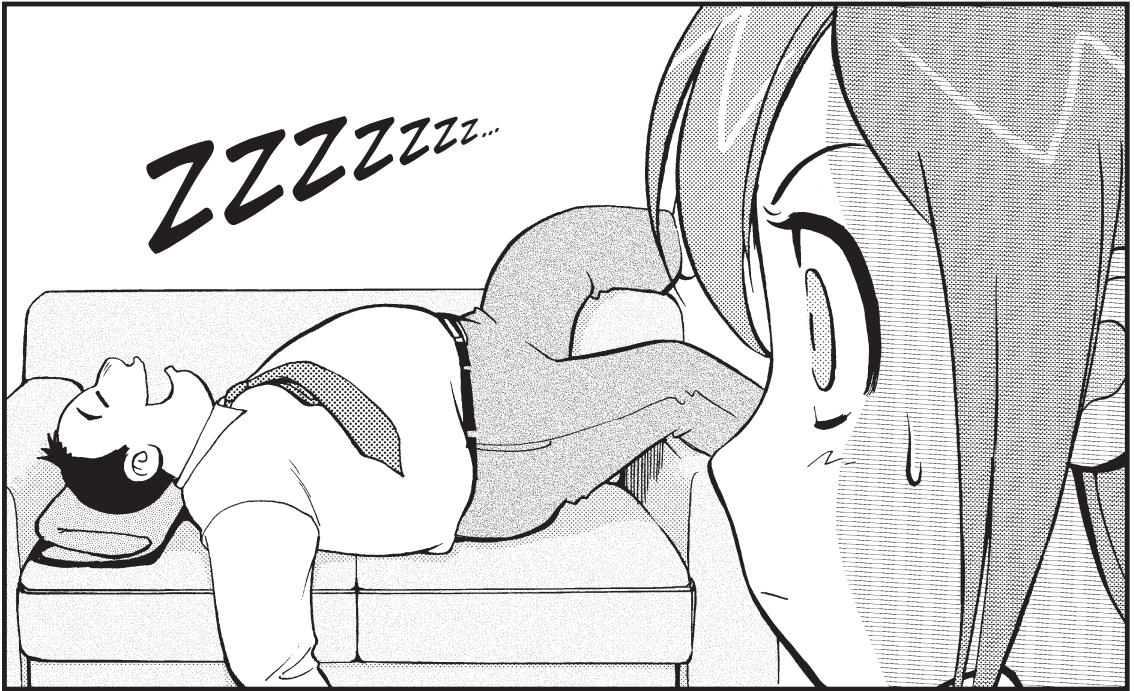
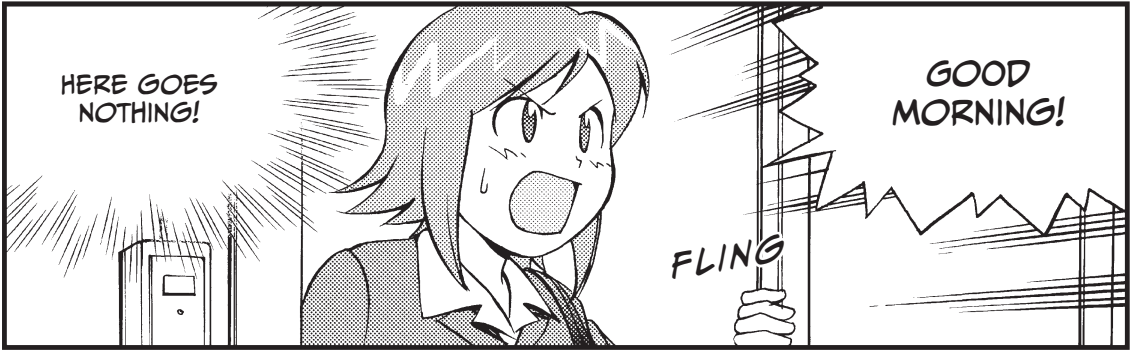
I'LL WORK HARD!!

THE ASAGAKE TIMES  
SANDA-CHO DISTRIBUTOR



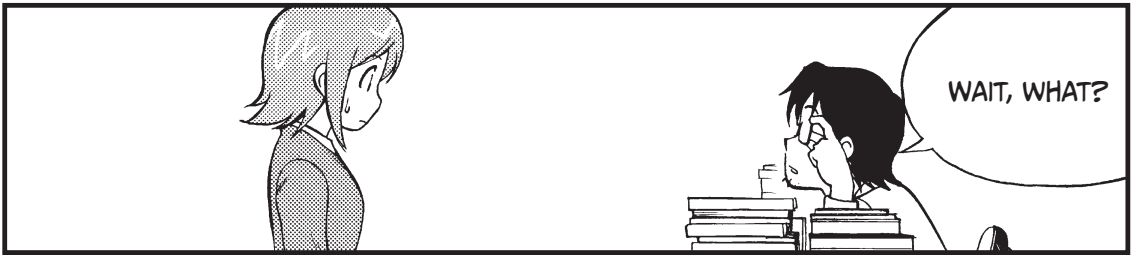
あさけ新聞  
算田町支局







WILL YOU  
LEAVE IT,  
PLEASE?



WAIT, WHAT?



OH, YOU HAVE  
BEEN ASSIGNED  
HERE TODAY.



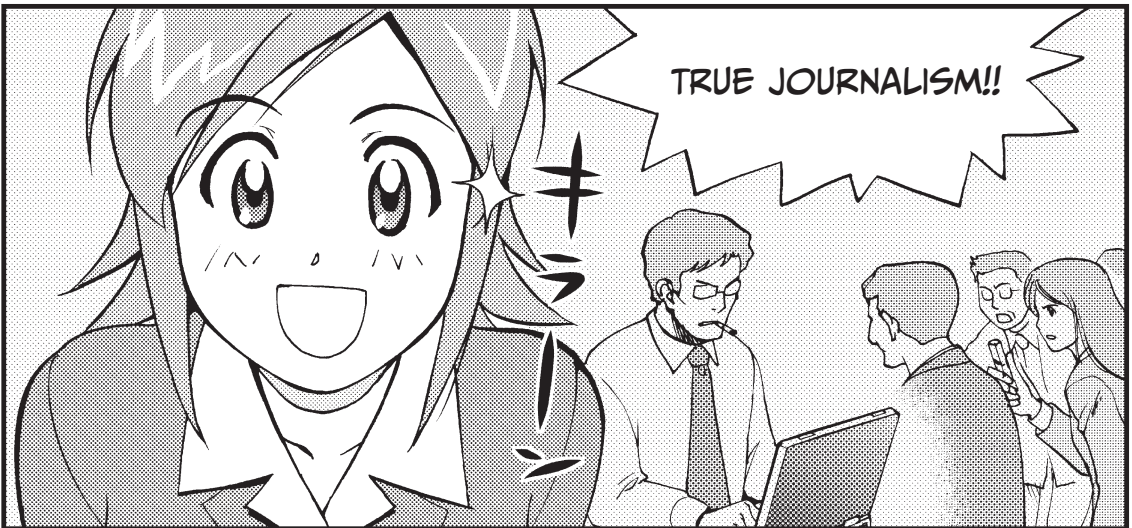
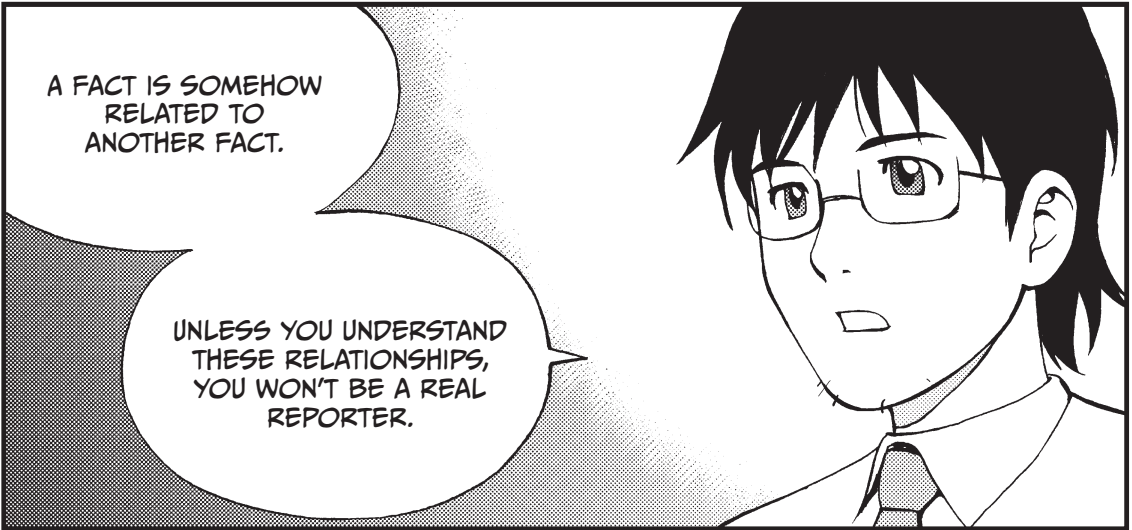
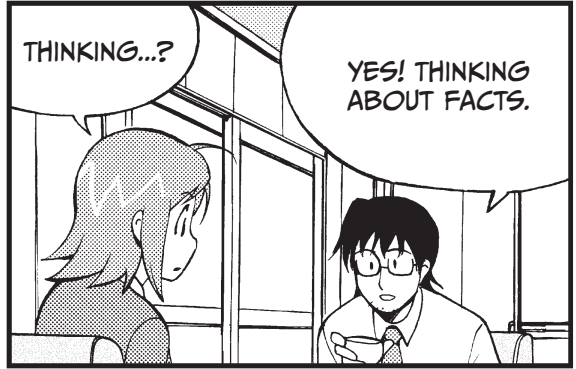
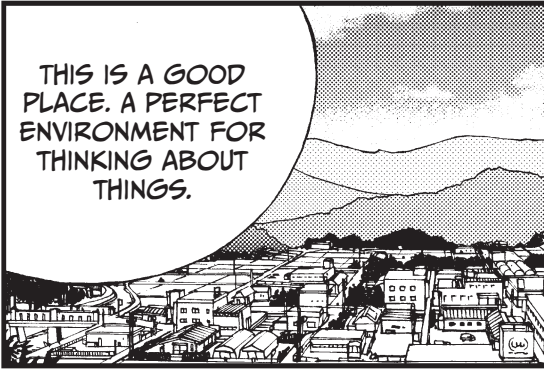
I'M NORIKO  
HIKIMA.

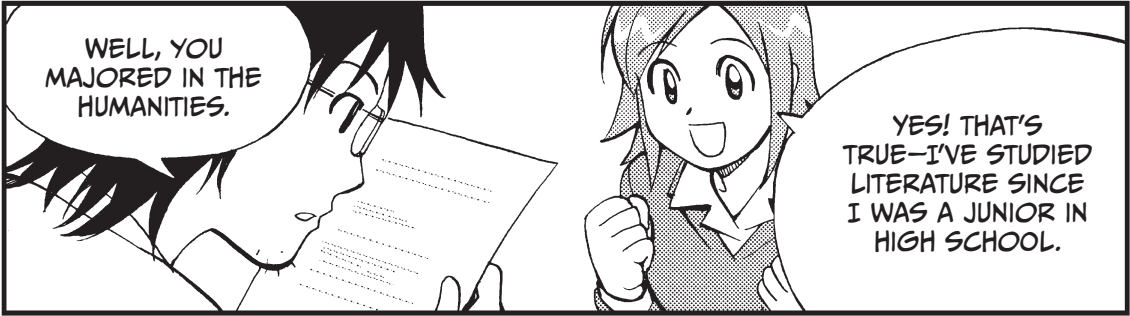
LONG TRIP,  
WASN'T IT? I'M  
KAKERU SEKI, THE  
HEAD OF THIS  
OFFICE.



THE BIG GUY THERE  
IS FUTOSHI MASUI,  
MY ONLY SOLDIER.

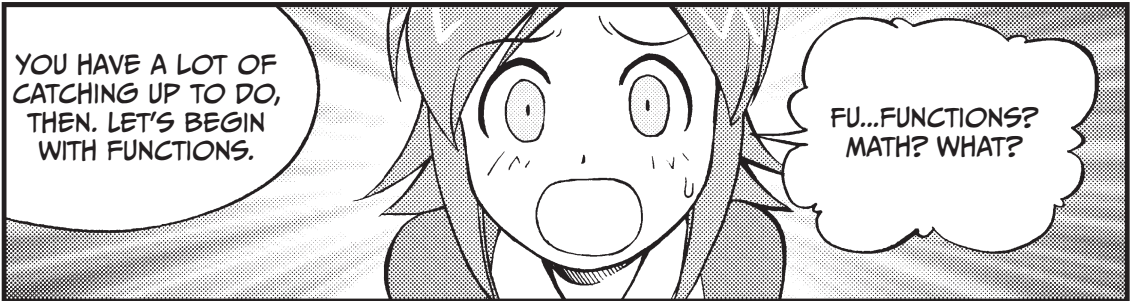
JUST  
TWO OF  
THEM...





WELL, YOU MAJORED IN THE HUMANITIES.

YES! THAT'S TRUE—I'VE STUDIED LITERATURE SINCE I WAS A JUNIOR IN HIGH SCHOOL.



YOU HAVE A LOT OF CATCHING UP TO DO, THEN. LET'S BEGIN WITH FUNCTIONS.

FU...FUNCTIONS? MATH? WHAT?



WHEN ONE THING CHANGES, IT INFLUENCES ANOTHER THING. A FUNCTION IS A CORRELATION.

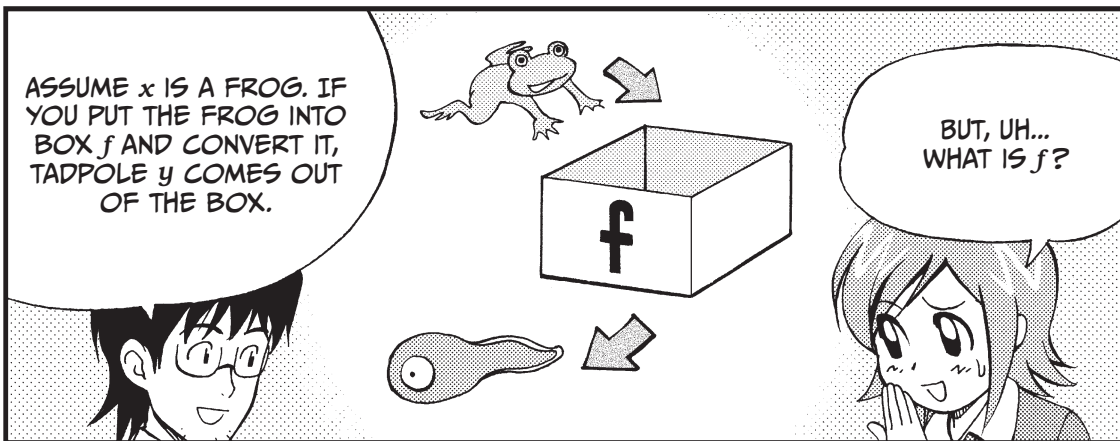
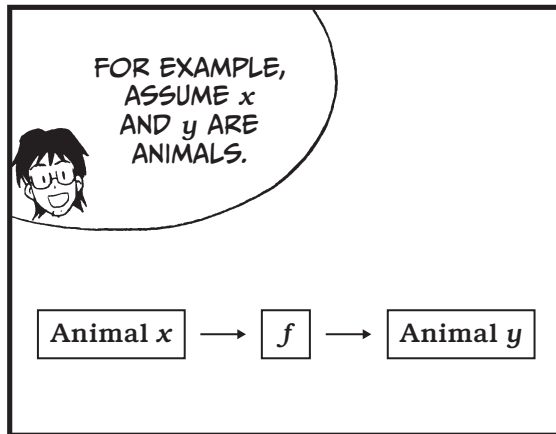
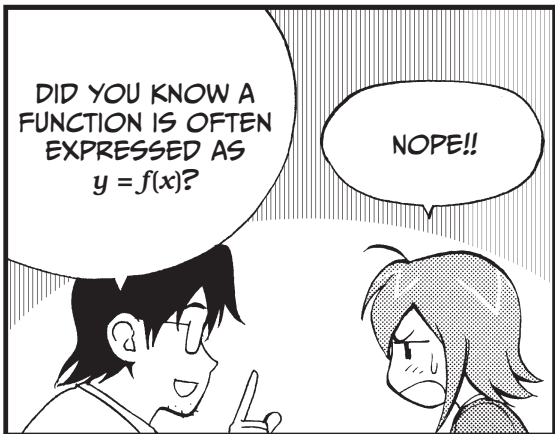
YOU CAN THINK OF THE WORLD ITSELF AS ONE BIG FUNCTION.

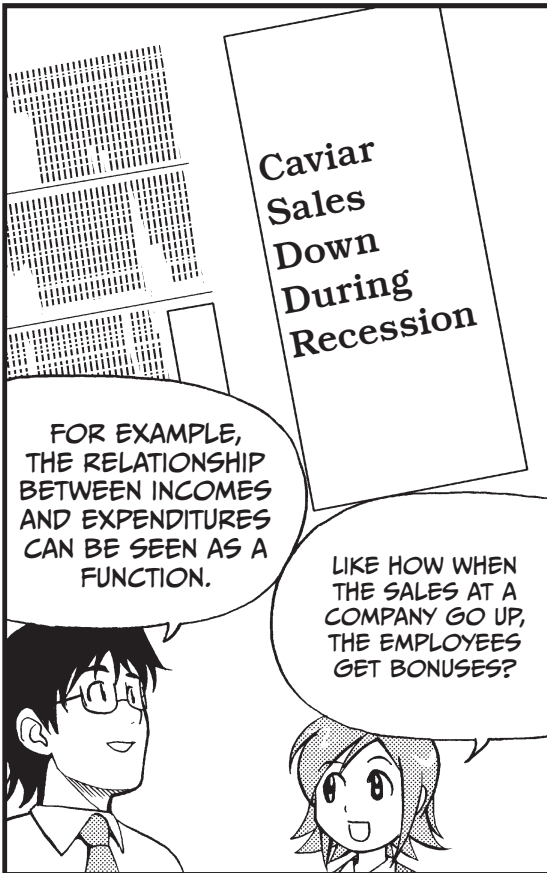
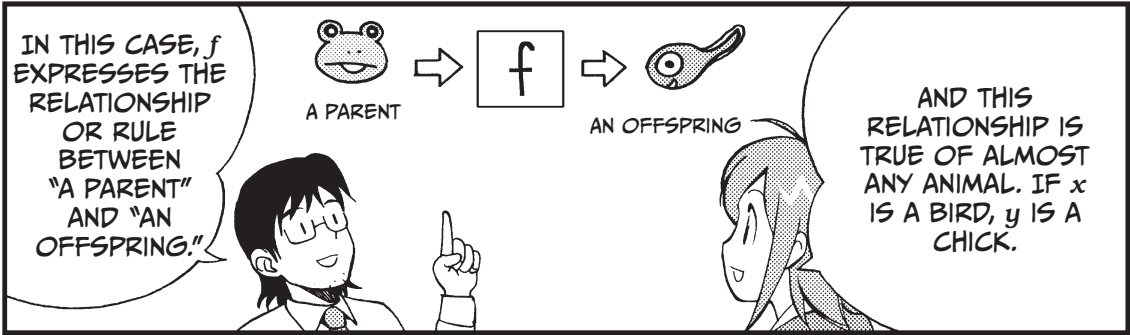


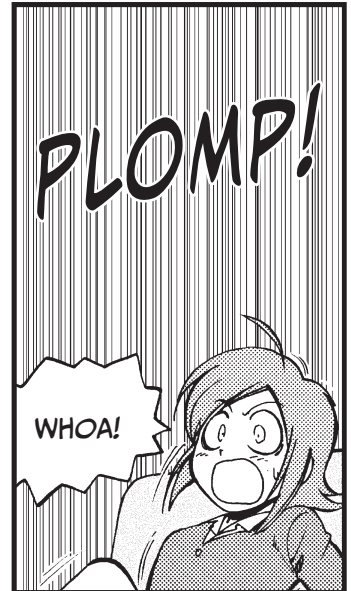
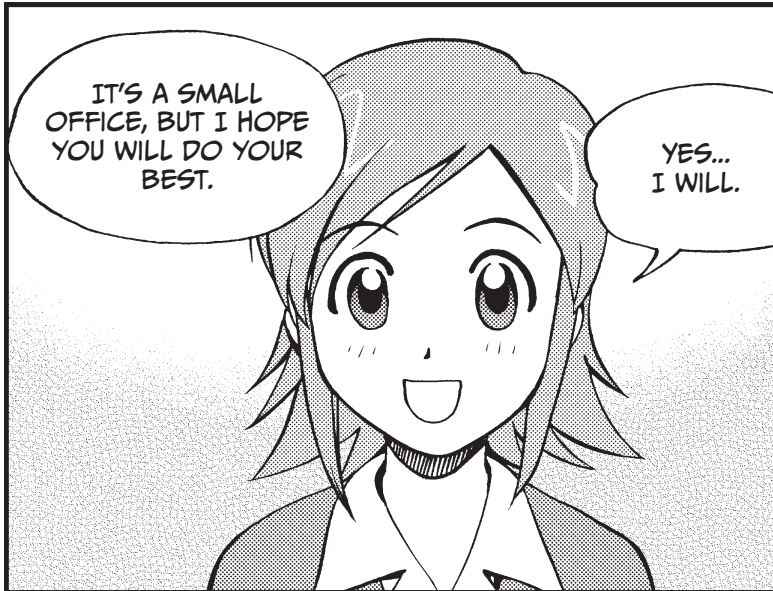
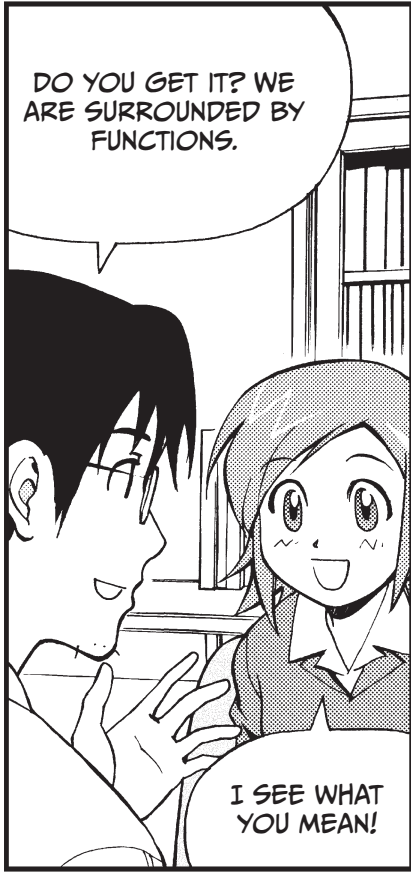
A FUNCTION DESCRIBES A RELATION, CAUSALITY, OR CHANGE.

AS JOURNALISTS, OUR JOB IS TO FIND THE REASON WHY THINGS HAPPEN—THE CAUSALITY.

YES...







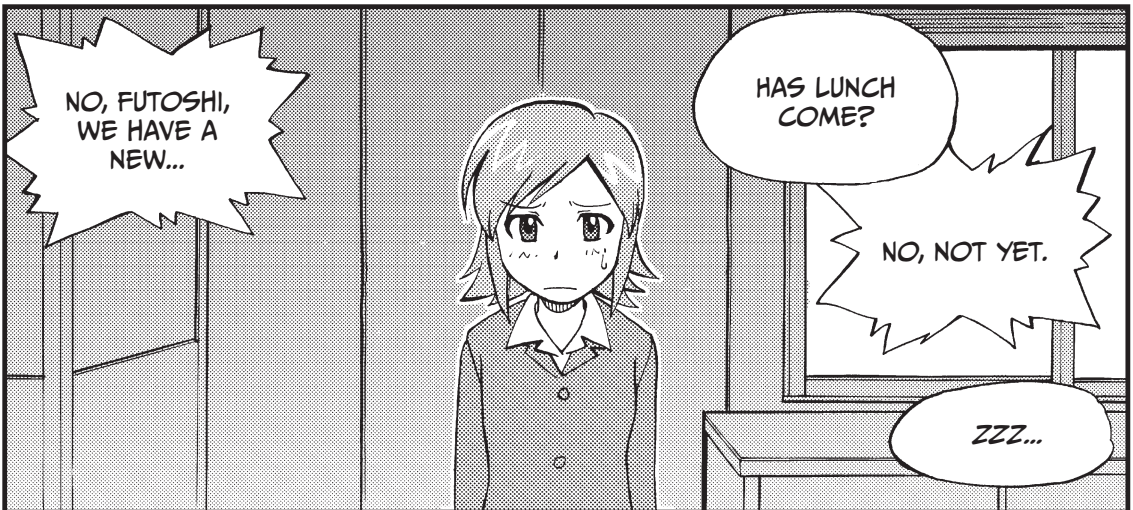
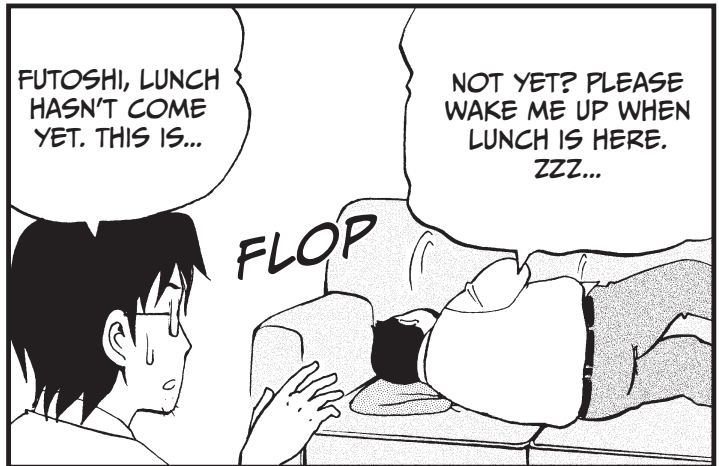
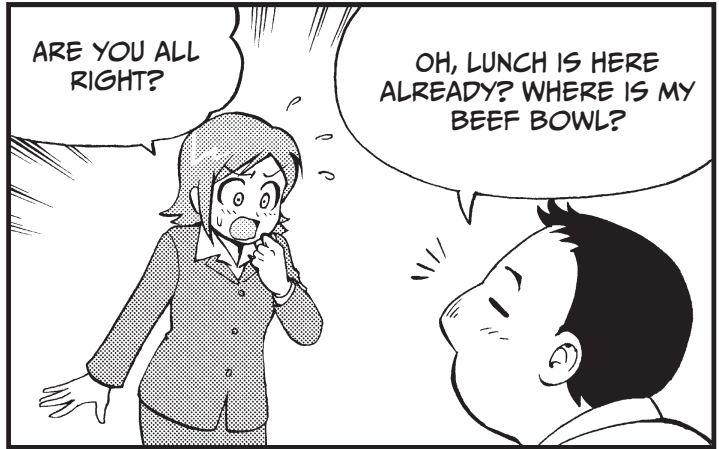
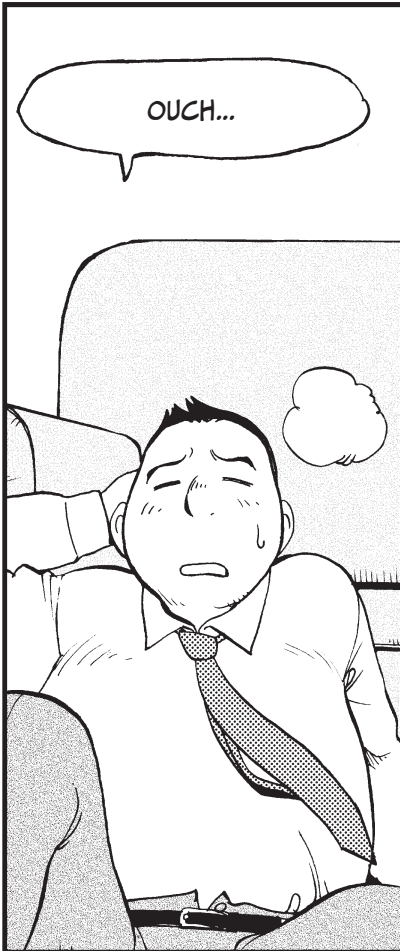
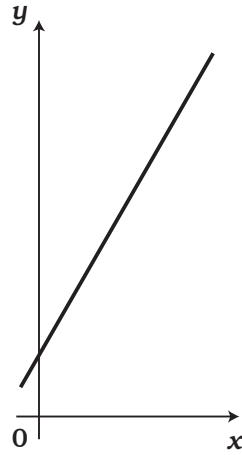
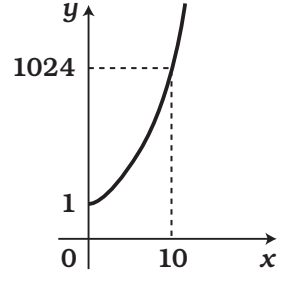


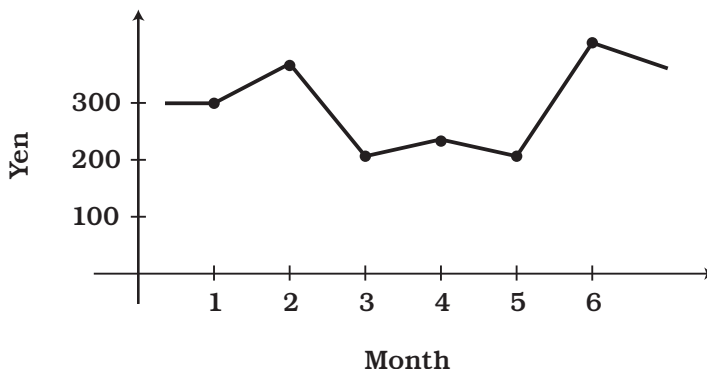
TABLE 1: CHARACTERISTICS OF FUNCTIONS

SUBJECT	CALCULATION	GRAPH
Causality	<p>The frequency of a cricket's chirp is determined by temperature. We can express the relationship between <math>y</math> chirps per minute of a cricket at temperature <math>x^\circ\text{C}</math> approximately as</p> $y = g(x) = 7x - 30$ <p style="text-align: center;"> <math>\uparrow</math>                      <math>\downarrow</math>  <math>x = 27^\circ</math>    <math>7 \times 27 - 30</math> </p> <p>The result is 159 chirps a minute.</p>	<p>When we graph these functions, the result is a straight line. That's why we call them linear functions.</p> 
Changes	<p>The speed of sound <math>y</math> in meters per second (m/s) in the air at <math>x^\circ\text{C}</math> is expressed as</p> $y = v(x) = 0.6x + 331$ <p>At <math>15^\circ\text{C}</math>,</p> $y = v(15) = 0.6 \times 15 + 331 = 340 \text{ m/s}$ <p>At <math>-5^\circ\text{C}</math>,</p> $y = v(-5) = 0.6 \times (-5) + 331 = 328 \text{ m/s}$	
Unit Conversion	<p>Converting <math>x</math> degrees Fahrenheit (<math>^\circ\text{F}</math>) into <math>y</math> degrees Celsius (<math>^\circ\text{C}</math>)</p> $y = f(x) = \frac{5}{9}(x - 32)$ <p>So now we know <math>50^\circ\text{F}</math> is equivalent to</p> $\frac{5}{9}(50 - 32) = 10^\circ\text{C}$	<p>The graph is an exponential function.</p> 
	<p>Computers store numbers using a binary system (1s and 0s). A binary number with <math>x</math> bits (or binary digits) has the potential to store <math>y</math> numbers.</p> $y = b(x) = 2^x$ <p>(This is described in more detail on page 131.)</p>	

THE GRAPHS OF SOME FUNCTIONS CANNOT BE EXPRESSED BY STRAIGHT LINES OR CURVES WITH A REGULAR SHAPE.



The stock price  $P$  of company A in month  $x$  in 2009 is  
 $y = P(x)$

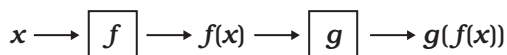


$P(x)$  cannot be expressed by a known function, but it is still a function. If you could find a way to predict  $P(7)$ , the stock price in July, you could make a big profit.

COMBINING TWO OR MORE FUNCTIONS IS CALLED "THE COMPOSITION OF FUNCTIONS." COMBINING FUNCTIONS ALLOWS US TO EXPAND THE RANGE OF CAUSALITY.



A composite function  
of  $f$  and  $g$



## EXERCISE

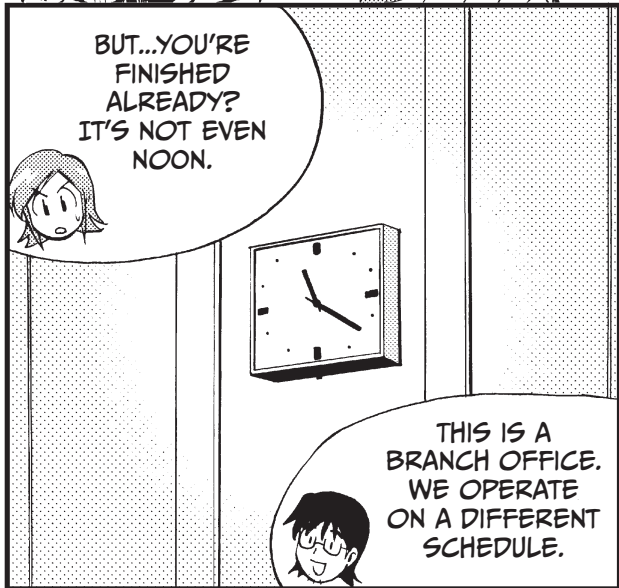
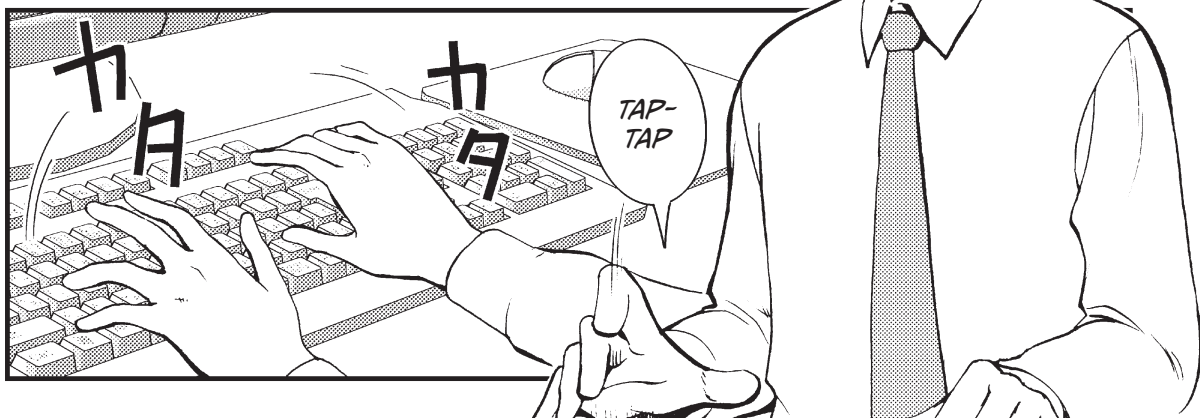
1. Find an equation that expresses the frequency of  $z$  chirps/minute of a cricket at  $x^\circ\text{F}$ .

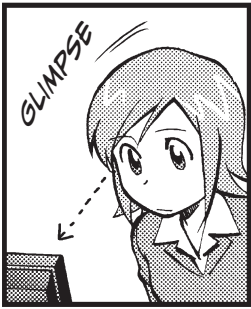
1

LET'S DIFFERENTIATE A FUNCTION!



# APPROXIMATING WITH FUNCTIONS





TO: EDITORS  
SUBJECT: TODAY'S HEADLINES

A BEAR RAMPAGES IN A HOUSE AGAIN—NO INJURIES  
THE REPUTATION OF SANDA-CHO WATERMELONS  
IMPROVES IN THE PREFECTURE

DO YOU...DO  
YOU ALWAYS  
FILE STORIES  
LIKE THIS?

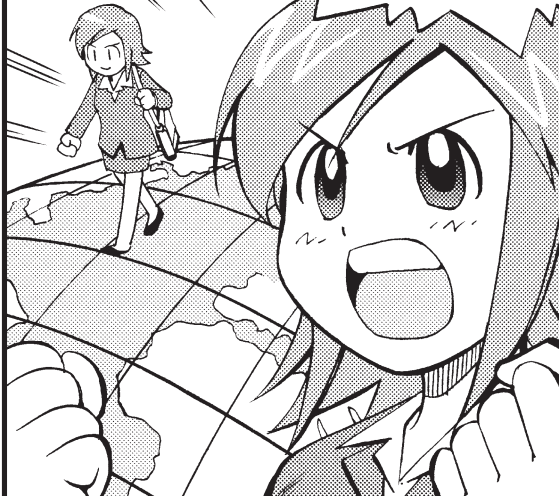


LOCAL NEWS LIKE  
THIS IS NOT BAD.  
BESIDES, HUMAN-  
INTEREST STORIES  
CAN BE...

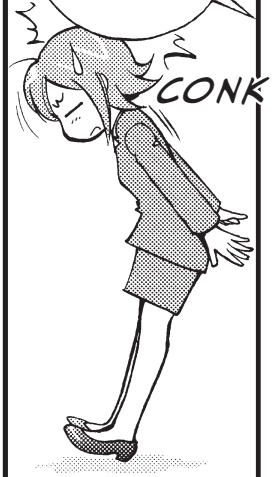


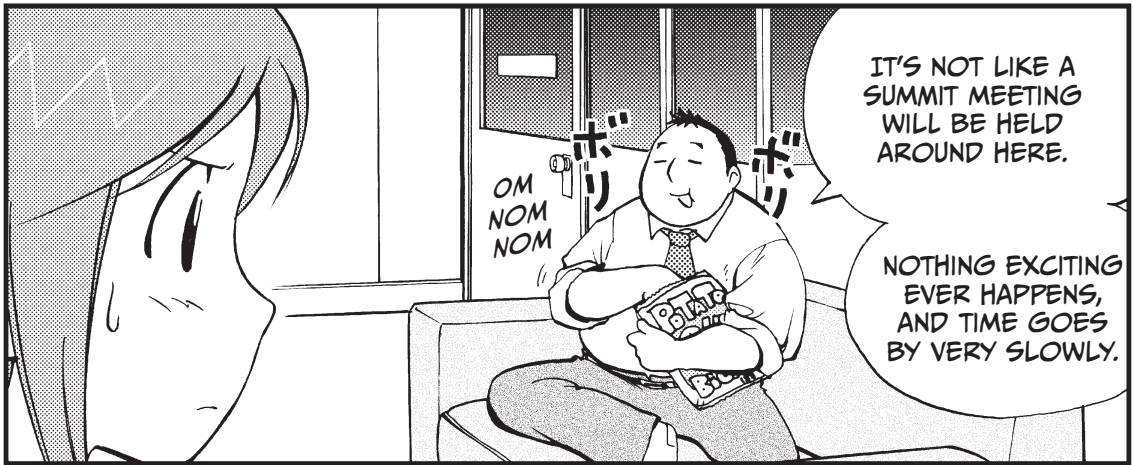
POLITICS, FOREIGN  
AFFAIRS, THE  
ECONOMY...

I WANT TO  
COVER THE  
HARD-HITTING  
ISSUES!!



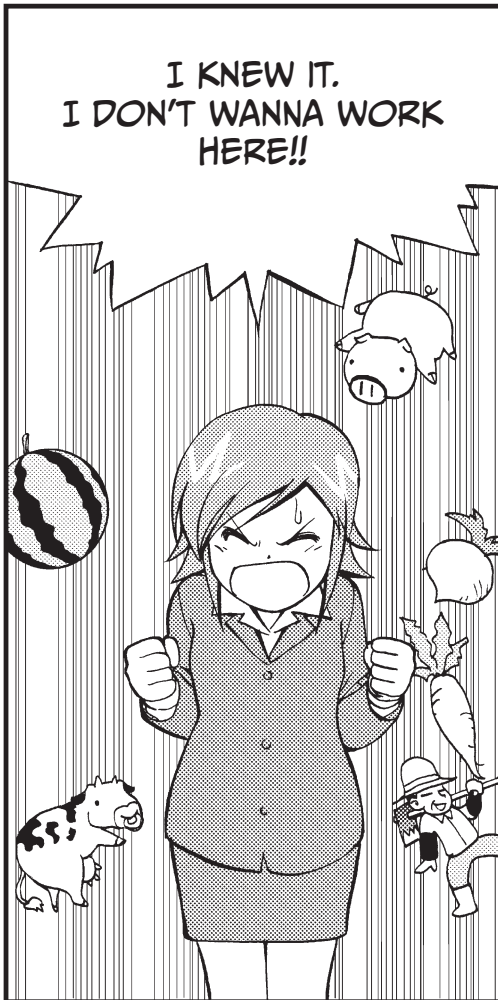
AH...THAT'S  
IMPOSSIBLE.





IT'S NOT LIKE A SUMMIT MEETING WILL BE HELD AROUND HERE.

NOTHING EXCITING EVER HAPPENS, AND TIME GOES BY VERY SLOWLY.



I KNEW IT. I DON'T WANNA WORK HERE!!

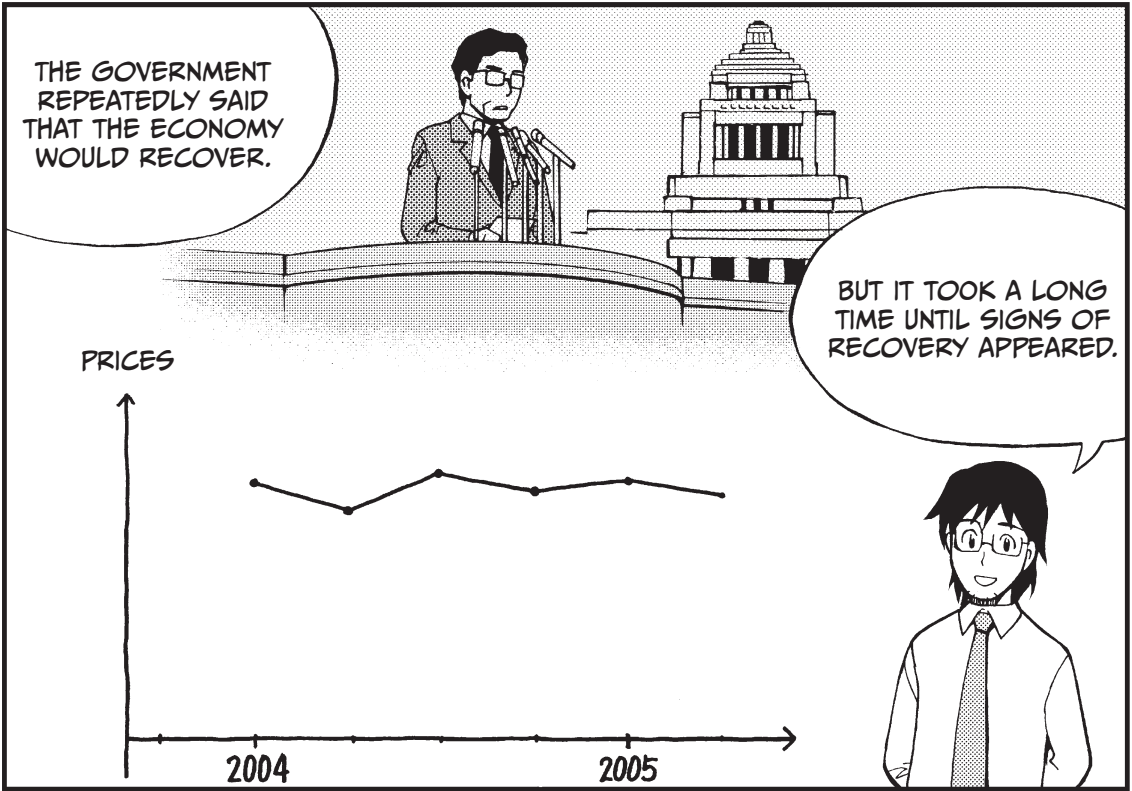


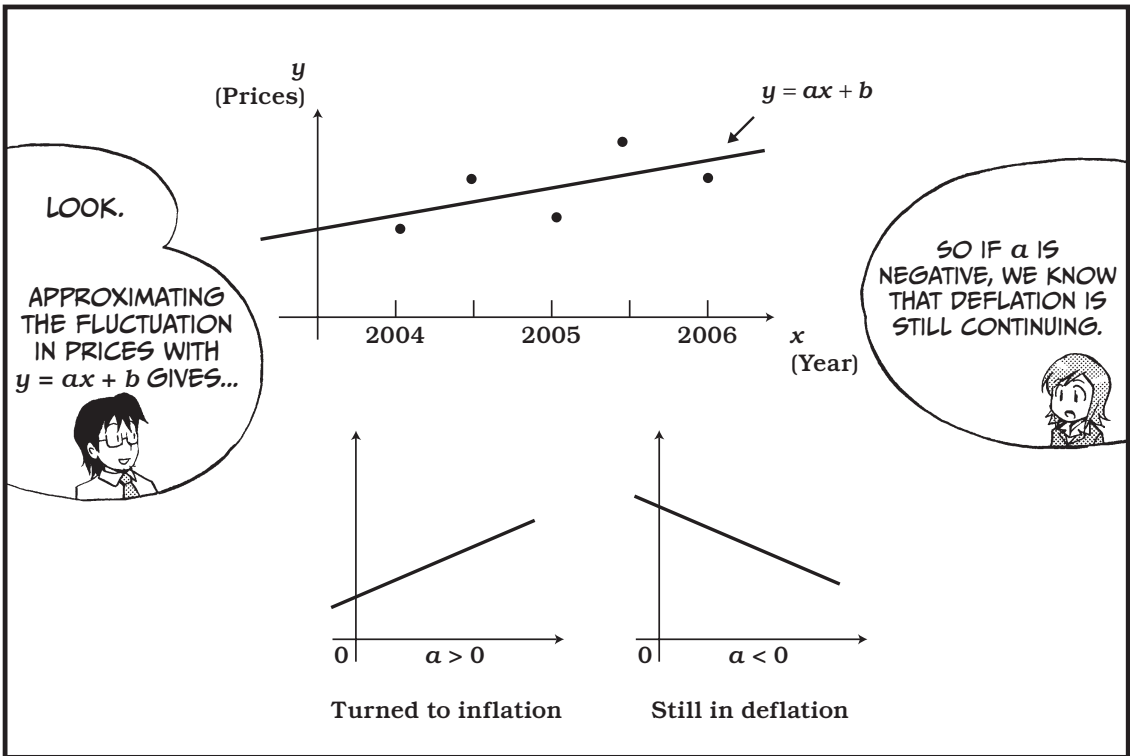
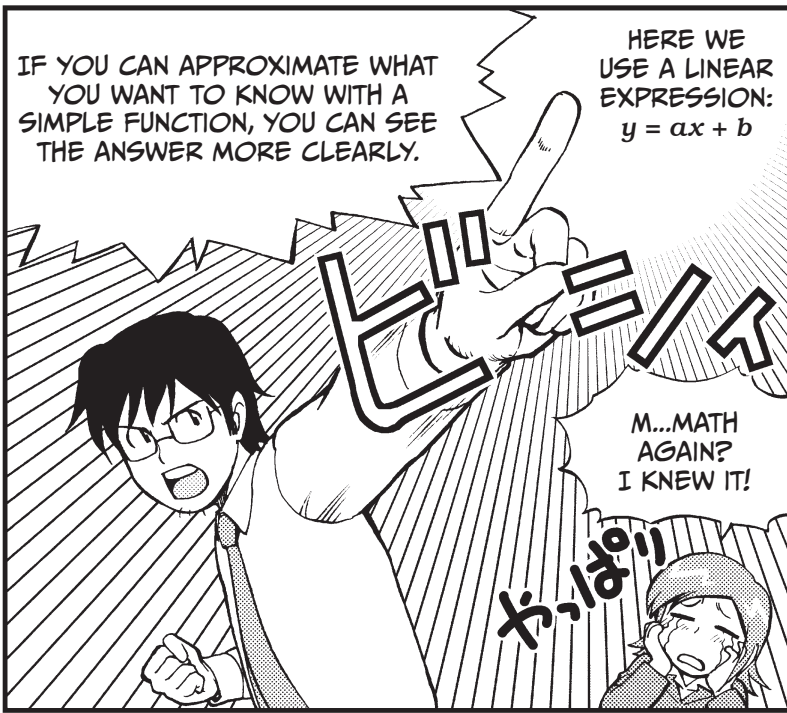
NORIKO, YOU CAN STILL BENEFIT FROM YOUR EXPERIENCES HERE.

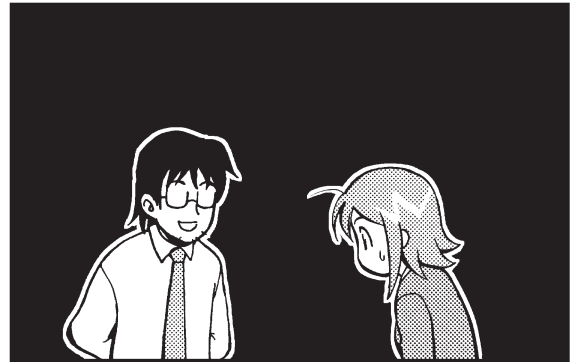
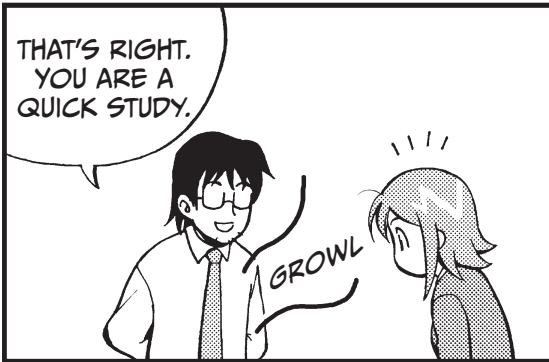


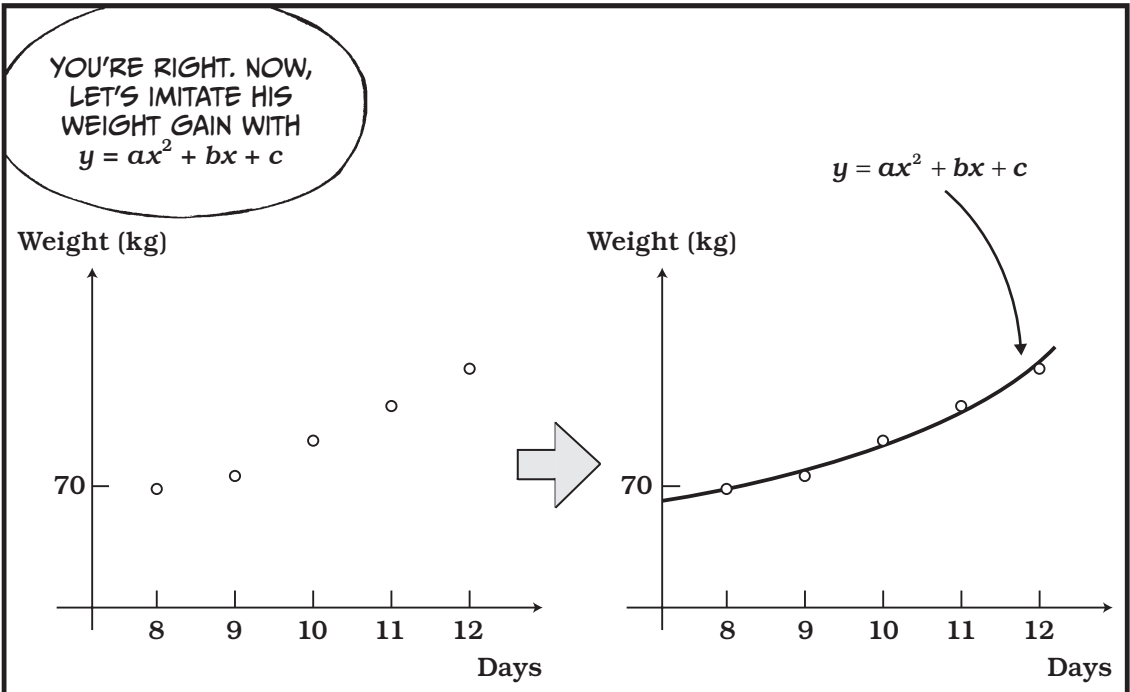
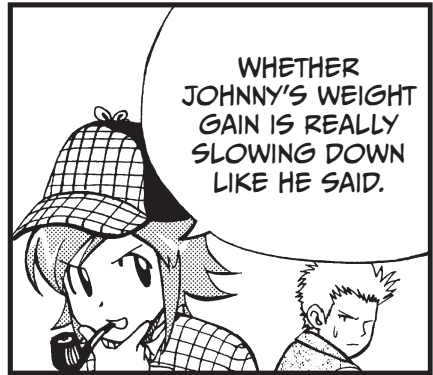
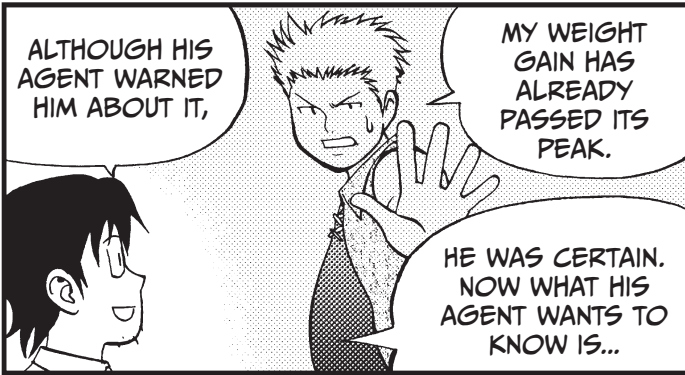
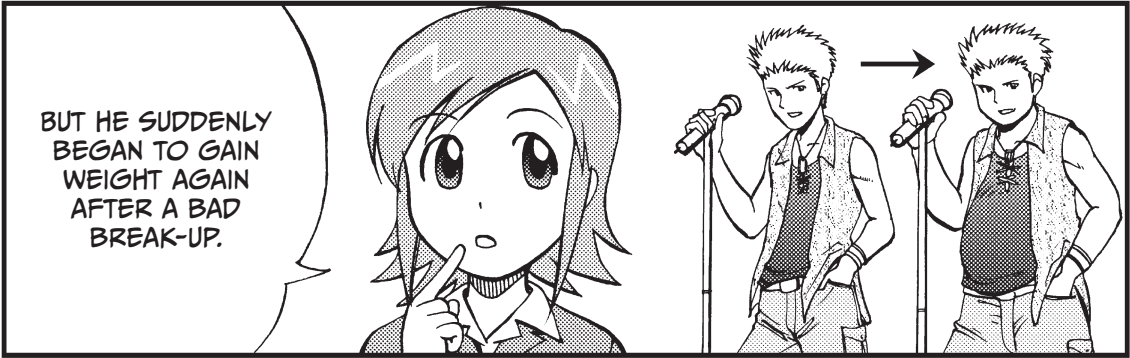
I DON'T KNOW WHICH BEAT YOU WANT TO COVER,

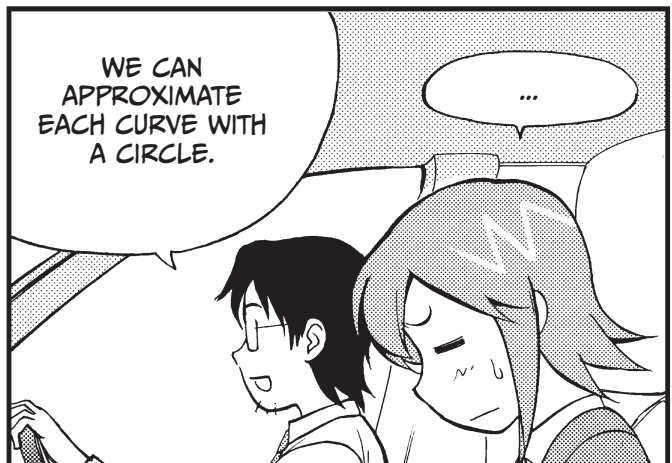
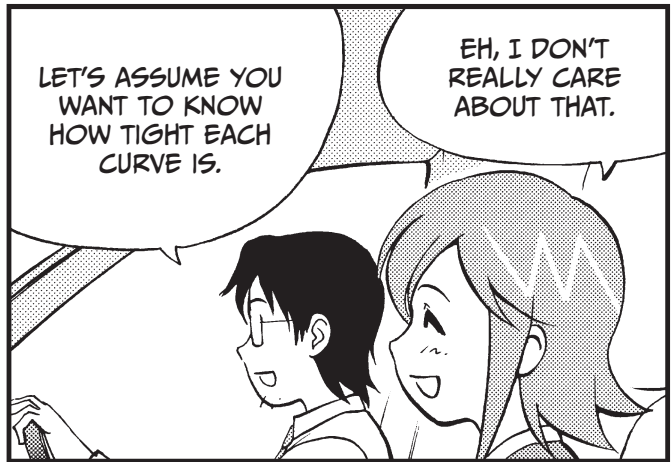
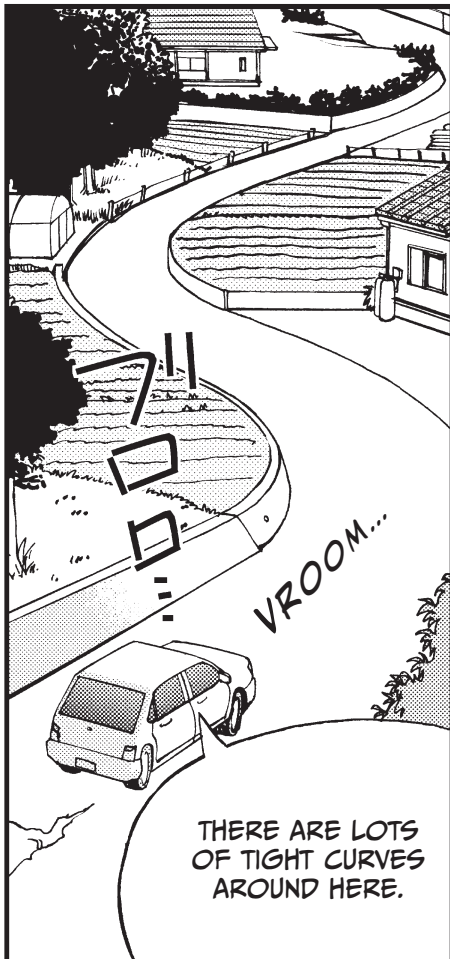
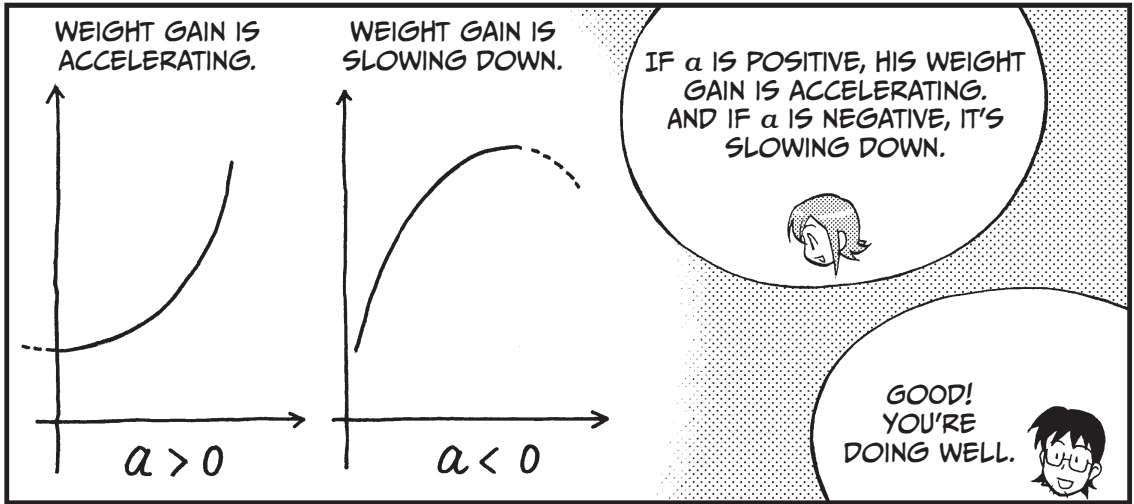
BUT I WILL TRAIN YOU WELL SO THAT YOU CAN BE ACCEPTED AT THE MAIN OFFICE.

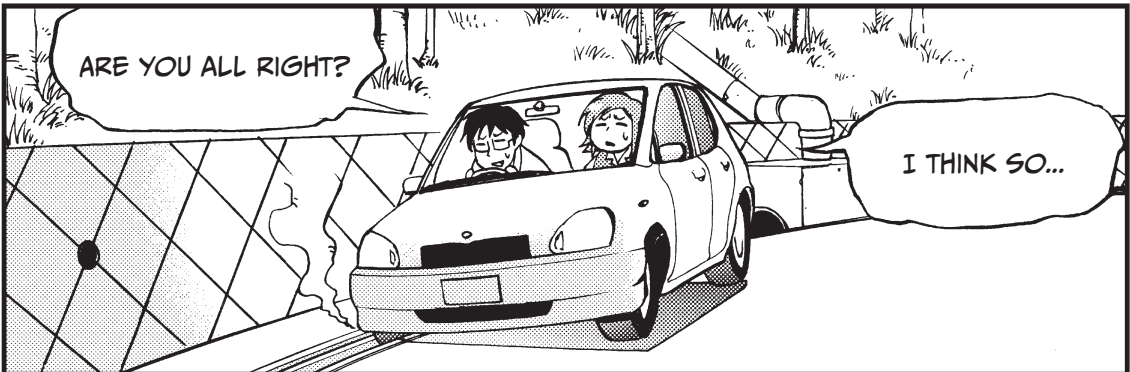
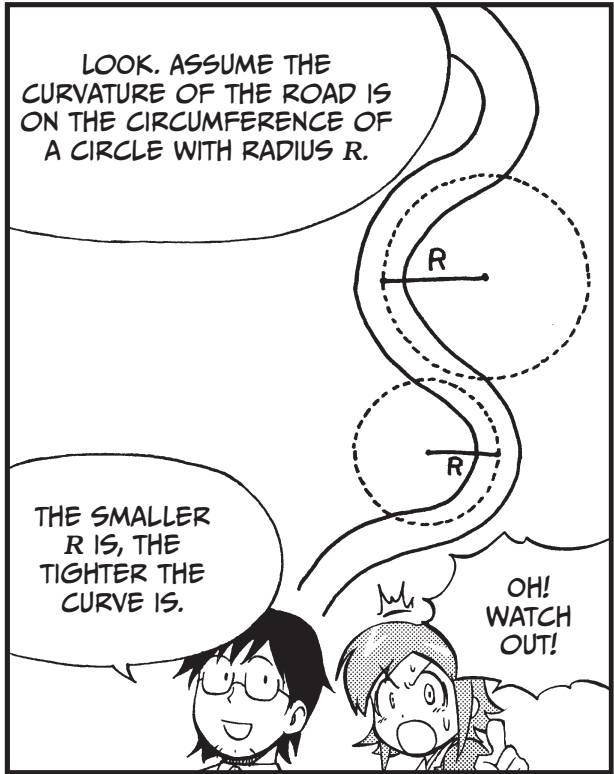
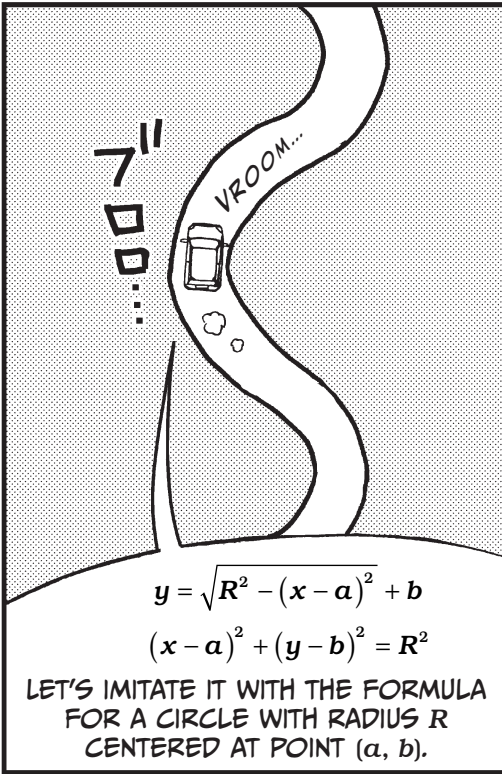


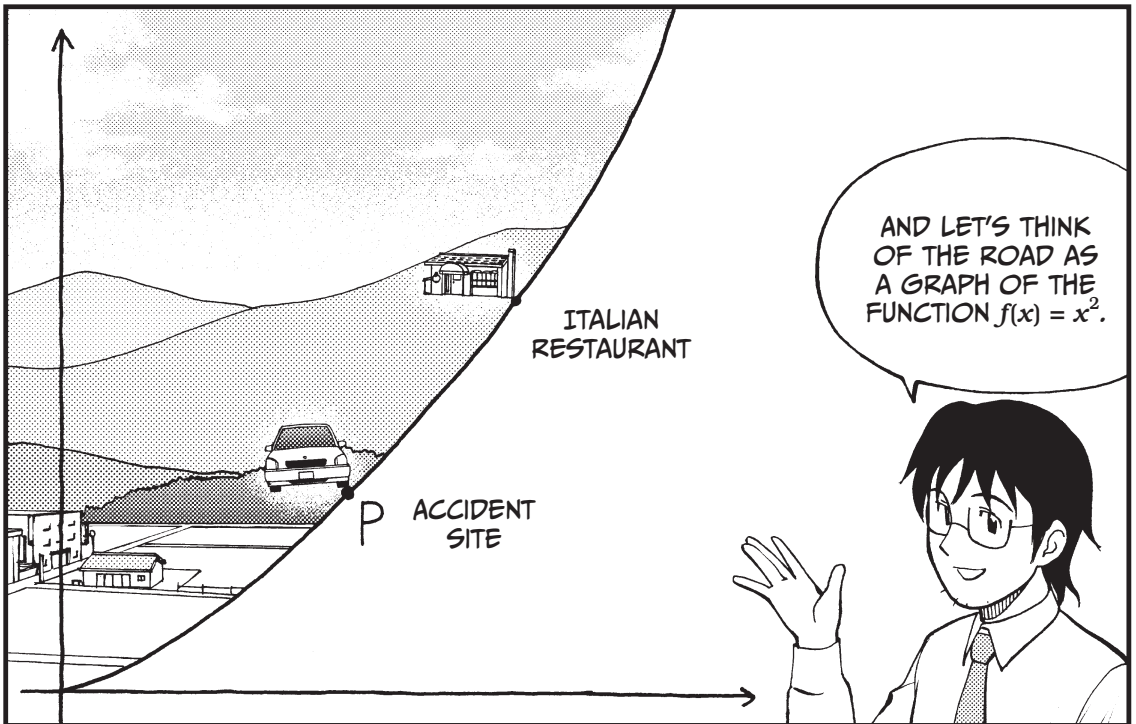
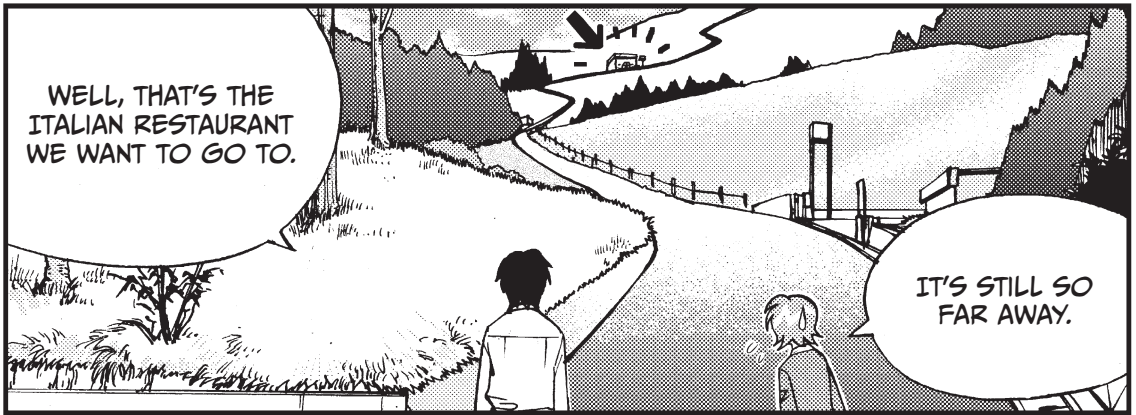


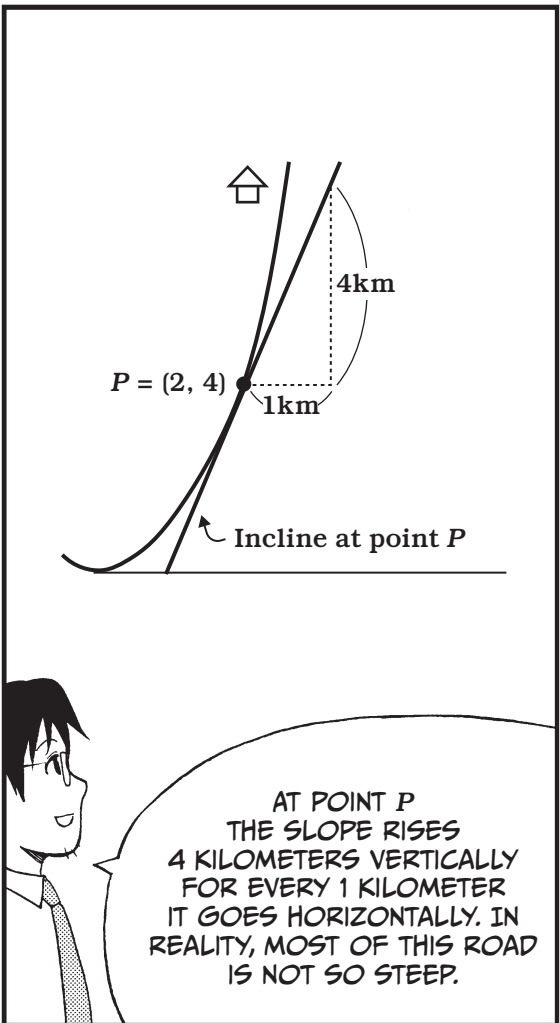
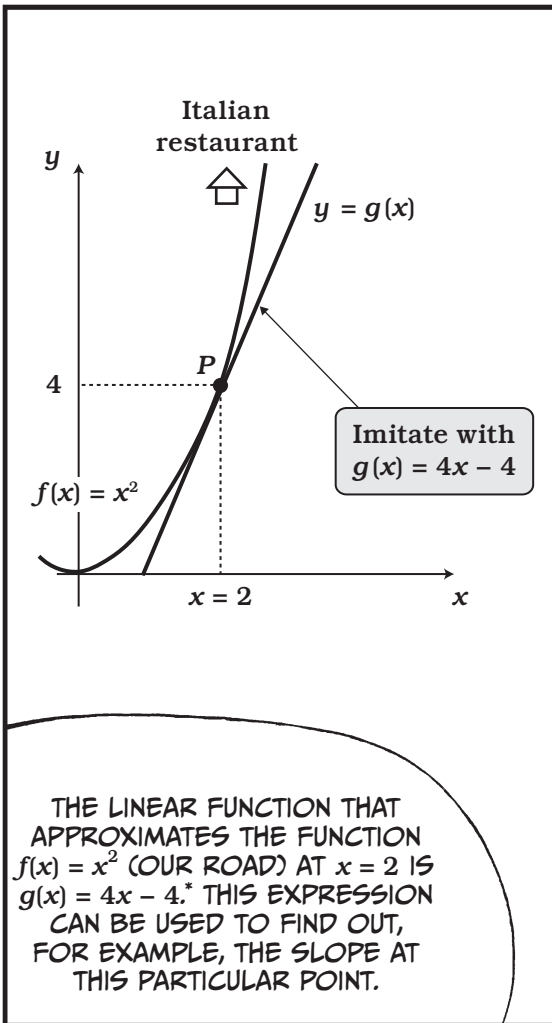












THE LINEAR FUNCTION THAT APPROXIMATES THE FUNCTION  $f(x) = x^2$  (OUR ROAD) AT  $x = 2$  IS  $g(x) = 4x - 4$ .<sup>\*</sup> THIS EXPRESSION CAN BE USED TO FIND OUT, FOR EXAMPLE, THE SLOPE AT THIS PARTICULAR POINT.

AT POINT P THE SLOPE RISES 4 KILOMETERS VERTICALLY FOR EVERY 1 KILOMETER IT GOES HORIZONTALLY. IN REALITY, MOST OF THIS ROAD IS NOT SO STEEP.

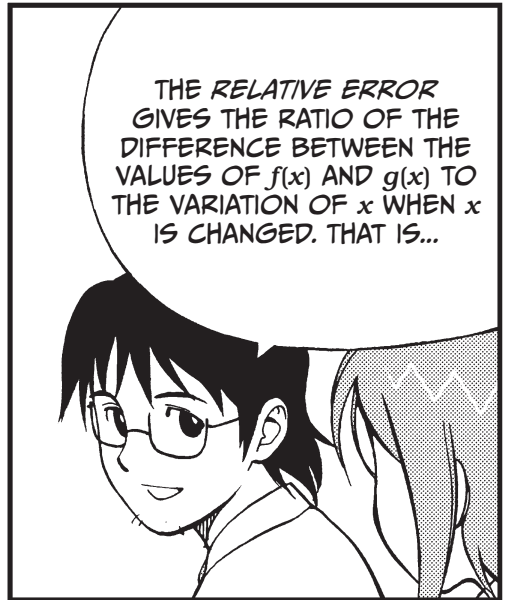
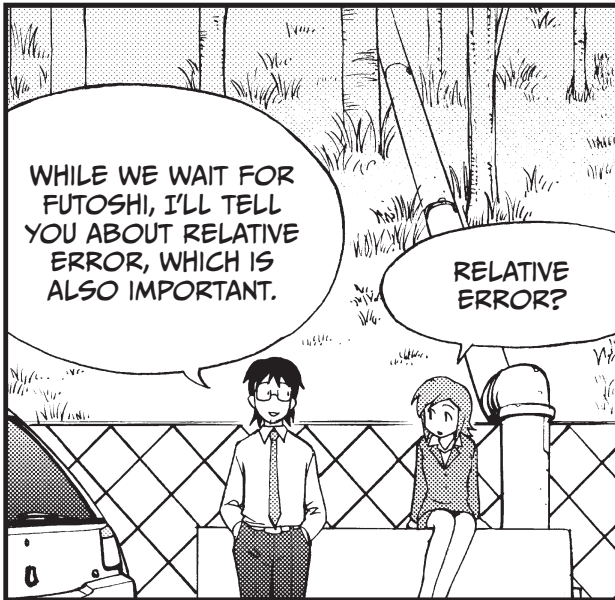
\* THE REASON IS GIVEN ON PAGE 39.

FUTOSHI? WE'VE HAD AN ACCIDENT. WILL YOU HELP US?

THE ACCIDENT SITE? IT'S POINT P.

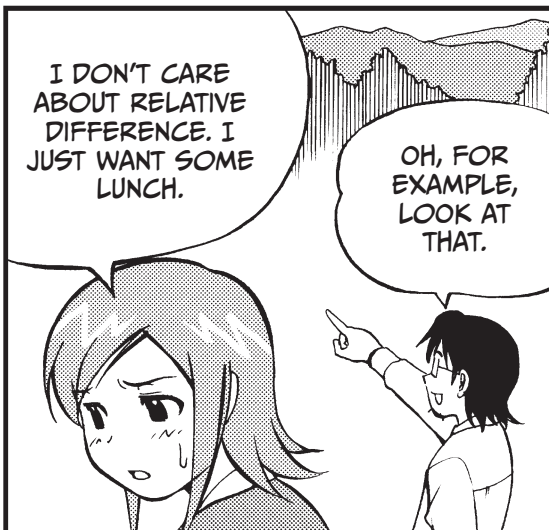
WHAT FUNCTION SHOULD I USE TO APPROXIMATE THE INSIDE OF YOUR HEAD?

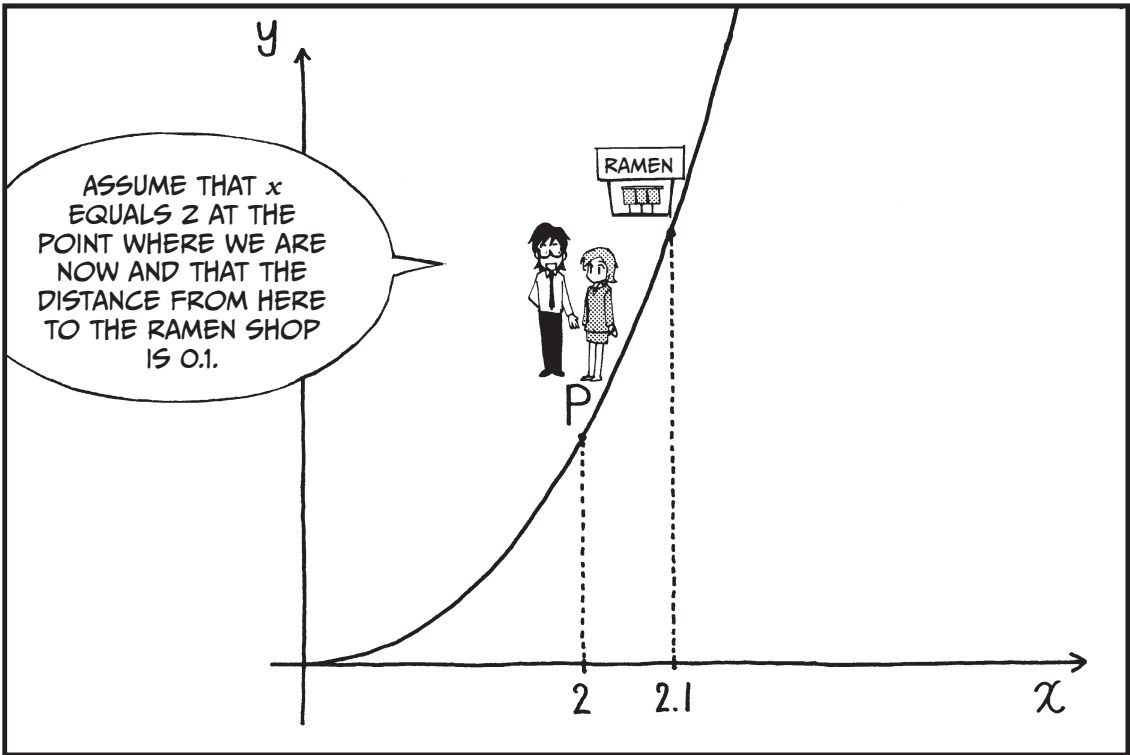
# CALCULATING THE RELATIVE ERROR



Our original function	Our approximating function
↓	↓
Difference between $f(x)$ and $g(x)$	
Relative error =	=
Change of $x$	

SIMPLE, RIGHT?



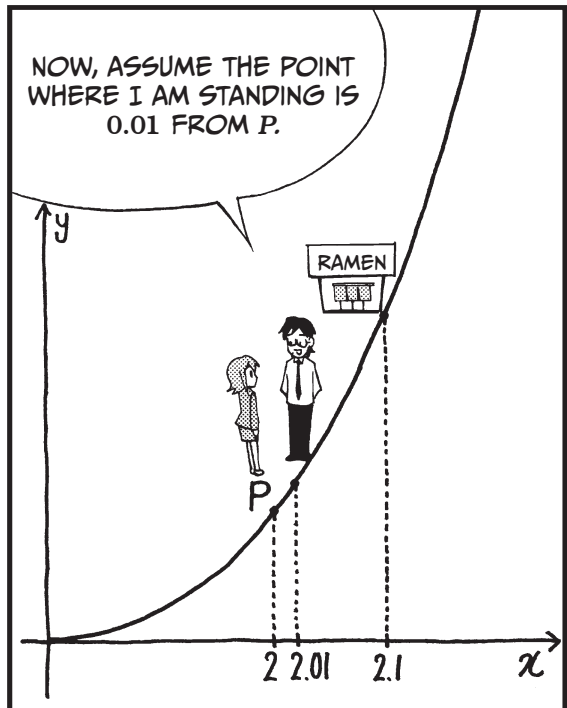


LET'S CHANGE  $x$  BY 0.1:  $x = 2$  BECOMES  $x = 2.1$ .

$$f(2.1) = 2.1^2 = 4.41$$

$$g(2.1) = 4 \times 2.1 - 4 = 4.4$$

SO THE DIFFERENCE IS  $f(2.1) - g(2.1) = 0.01$ , AND THE RELATIVE ERROR IS  $0.01 / 0.1 = 0.1$  (10 PERCENT).



CHANGE  $x$  BY 0.01:  $x = 2$   
BECOMES  $x = 2.01$ .

$$\text{ERROR } f(2.01) - g(2.01) = 4.0401 - 4.04 = 0.0001$$

RELATIVE ERROR

$$\frac{0.0001}{0.01} = 0.01$$

$$= [1\%]$$

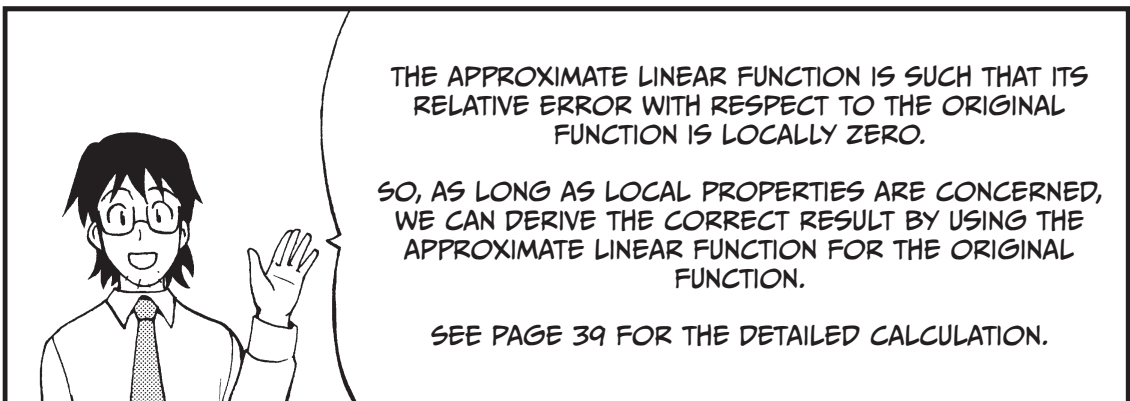
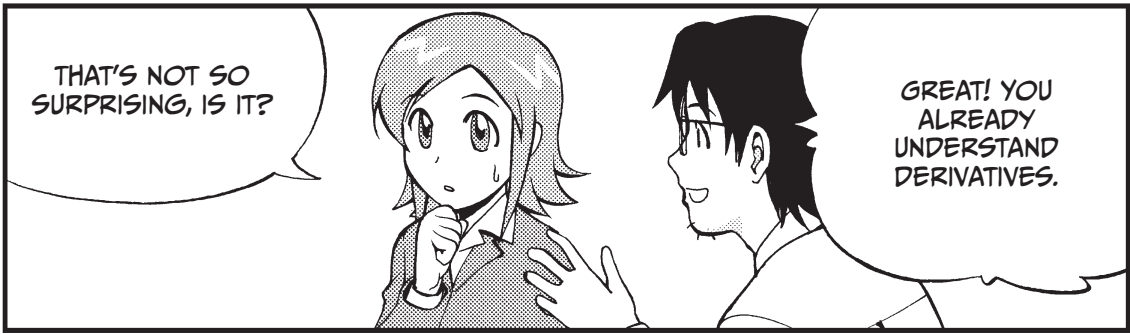
THE RELATIVE ERROR  
FOR THIS POINT IS  
SMALLER THAN FOR  
THE RAMEN SHOP.

IN OTHER WORDS, THE  
CLOSER I STAND TO  
THE ACCIDENT SITE, THE  
BETTER  $g(x)$  IMITATES  $f(x)$ .



As the variation approaches 0, the relative error also approaches 0.

Variation of $x$ from 2	$f(x)$	$g(x)$	Error	Relative error
1	9	8	1	100.0%
0.1	4.41	4.4	0.01	10.0%
0.01	4.0401	4.04	0.0001	1.0%
0.001	4.004001	4.004	0.000001	0.1%
⋮ ↓				⋮ ↓
0				0



RAMEN SANDA

ラーメン 箕田

ラーメン  
5  
あめ

SLURP

WHY IS FUTOSHI EATING SO MUCH? HE JUST CAME TO RESCUE US.

SIGH. I LIKE RAMEN, BUT I WANTED TO EAT ITALIAN FOOD.

NORIKO, WE CAN ALSO ESTIMATE THE COST-EFFECTIVENESS OF TV COMMERCIALS USING APPROXIMATE FUNCTIONS.

REALLY?

# THE DERIVATIVE IN ACTION!



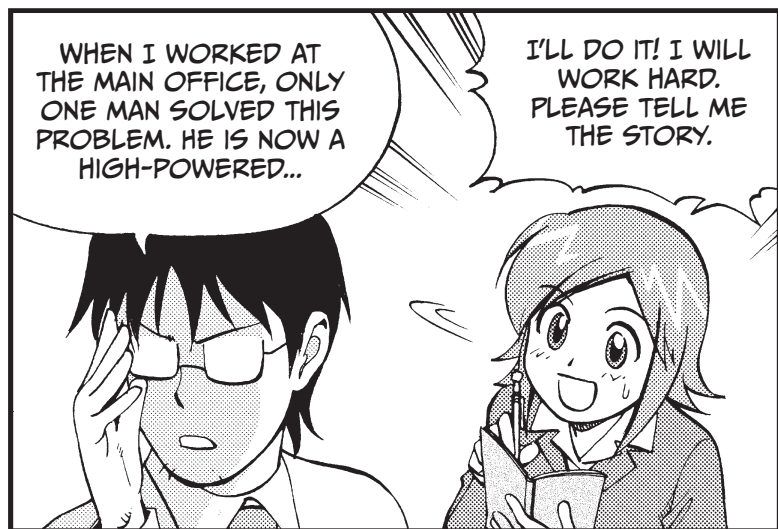
YOU KNOW THE BEVERAGE MANUFACTURER AMALGAMATED COLA?

LET'S CONSIDER WHETHER ONE OF THEIR EXECUTIVES INCREASED OR DECREASED THE AIRTIME OF THE COMPANY'S TV COMMERCIAL TO RAISE THE PROFIT FROM ITS POPULAR PRODUCTS.



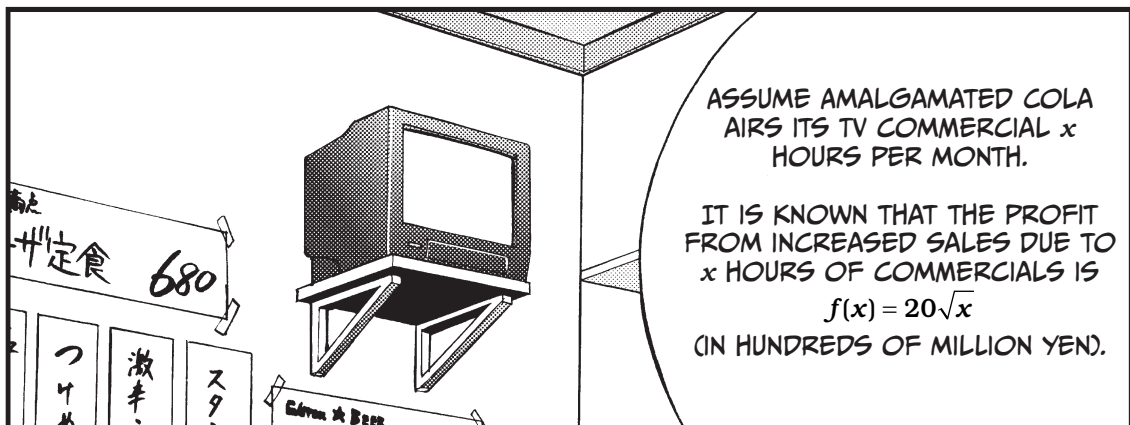
OKAY, I GUESS.

YOU KNOW...



WHEN I WORKED AT THE MAIN OFFICE, ONLY ONE MAN SOLVED THIS PROBLEM. HE IS NOW A HIGH-POWERED...

I'LL DO IT! I WILL WORK HARD. PLEASE TELL ME THE STORY.



ASSUME AMALGAMATED COLA AIRS ITS TV COMMERCIAL  $x$  HOURS PER MONTH.

IT IS KNOWN THAT THE PROFIT FROM INCREASED SALES DUE TO  $x$  HOURS OF COMMERCIALS IS

$$f(x) = 20\sqrt{x}$$

(IN HUNDREDS OF MILLION YEN).

AMALGAMATED COLA  
NOW AIRS THE TV  
COMMERCIAL FOR  
4 HOURS PER MONTH.



AND SINCE  
 $f(4) = 20\sqrt{4} = 40$ , THE  
COMPANY MAKES A PROFIT  
OF 4 BILLION YEN.

THE FEE FOR THE  
TV COMMERCIAL IS  
10 MILLION YEN PER  
MINUTE.

1-MINUTE COMMERCIAL =  
¥10 MILLION

T...TEN MILLION  
YEN!?

$f(x) = 20\sqrt{x}$  HUNDRED MILLION YEN

1-MIN COMMERCIAL = ¥10 MILLION

NOW, A NEWLY  
APPOINTED EXECUTIVE  
HAS DECIDED TO  
RECONSIDER THE  
AIRTIME OF THE TV  
COMMERCIAL. DO YOU  
THINK HE WILL INCREASE  
THE AIRTIME OR  
DECREASE IT?



HMM.



## STEP 1

SINCE  $f(x) = 20\sqrt{x}$  HUNDRED MILLION YEN IS A COMPLICATED FUNCTION, LET'S MAKE A SIMILAR LINEAR FUNCTION TO ROUGHLY ESTIMATE THE RESULT.

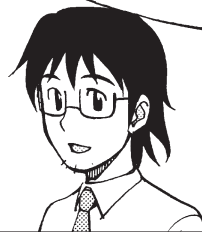
$$f(x) = 20\sqrt{x}$$

HUNDRED MILLION YEN

↓ IMITATE

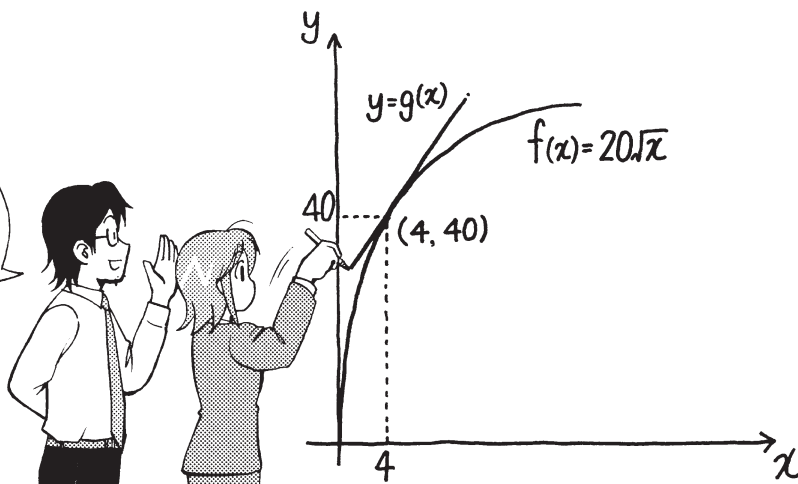
$$y = g(x)$$

SINCE IT'S IMPOSSIBLE TO IMITATE THE WHOLE FUNCTION WITH A LINEAR FUNCTION, WE WILL IMITATE IT IN THE VICINITY OF THE CURRENT AIRTIME OF  $x = 4$ .



## STEP 2

WE WILL DRAW A TANGENT LINE\* TO THE GRAPH OF  $f(x) = 20\sqrt{x}$  AT POINT  $(4, 40)$ .



\* Here is the calculation of the tangent line. (See also the explanation of the derivative on page 39.)

For  $f(x) = 20\sqrt{x}$ ,  $f'(4)$  is given as follows.

$$\begin{aligned} \frac{f(4+\varepsilon) - f(4)}{\varepsilon} &= \frac{20\sqrt{4+\varepsilon} - 20 \times 2}{\varepsilon} = 20 \frac{(\sqrt{4+\varepsilon} - 2) \times (\sqrt{4+\varepsilon} + 2)}{\varepsilon \times (\sqrt{4+\varepsilon} + 2)} \\ &= 20 \frac{4 + \varepsilon - 4}{\varepsilon(\sqrt{4+\varepsilon} + 2)} = \frac{20}{\sqrt{4+\varepsilon} + 2} \quad \textcircled{1} \end{aligned}$$

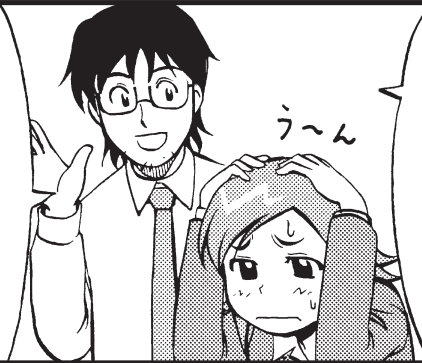
When  $\varepsilon$  approaches 0, the denominator of  $\textcircled{1}$   $\sqrt{4+\varepsilon} + 2 \rightarrow 4$ .

Therefore,  $\textcircled{1} \rightarrow 20 \div 4 = 5$ .

Thus, the approximate linear function  $g(x) = 5(x - 4) + 40 = 5x + 20$

IF THE CHANGE IN  $x$  IS LARGE—FOR EXAMPLE, AN HOUR—THEN  $g(x)$  DIFFERS FROM  $f(x)$  TOO MUCH AND CANNOT BE USED.

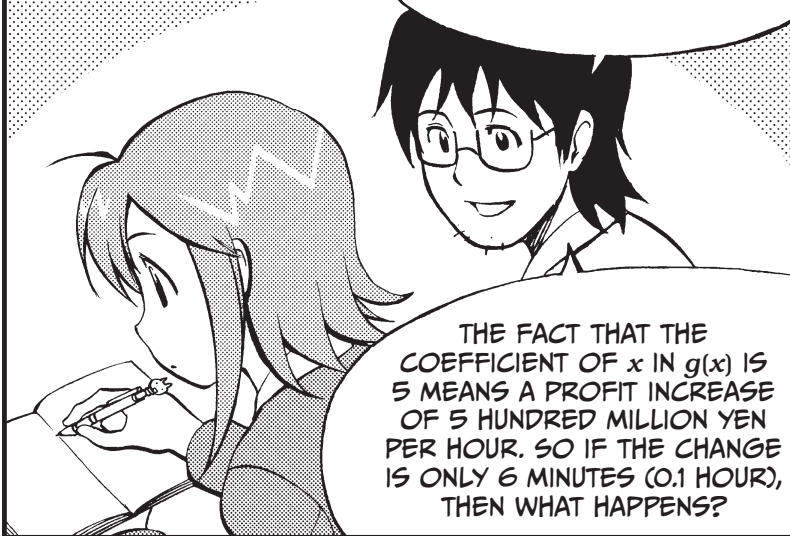
IN REALITY, THE CHANGE IN AIRTIME OF THE TV COMMERCIAL MUST ONLY BE A SMALL AMOUNT, EITHER AN INCREASE OR A DECREASE.



IF YOU CONSIDER AN INCREASE OR DECREASE OF, FOR EXAMPLE, 6 MINUTES (0.1 HOUR), THIS APPROXIMATION CAN BE USED, BECAUSE THE RELATIVE ERROR IS SMALL WHEN THE CHANGE IN  $x$  IS SMALL.

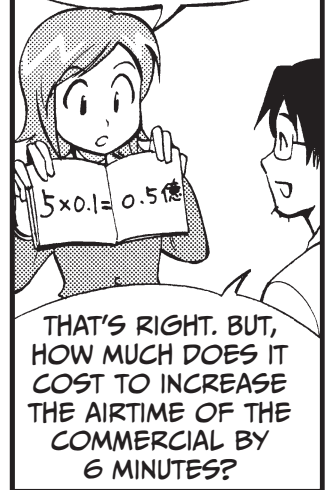
### STEP 3

IN THE VICINITY OF  $x = 4$  HOURS,  $f(x)$  CAN BE SAFELY APPROXIMATED AS ROUGHLY  $g(x) = 5x + 20$ .



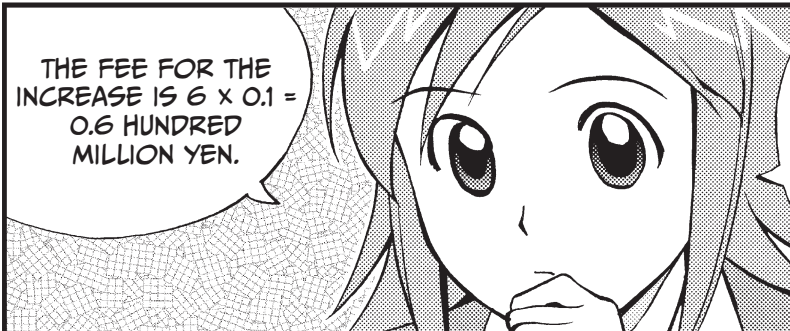
THE FACT THAT THE COEFFICIENT OF  $x$  IN  $g(x)$  IS 5 MEANS A PROFIT INCREASE OF 5 HUNDRED MILLION YEN PER HOUR. SO IF THE CHANGE IS ONLY 6 MINUTES (0.1 HOUR), THEN WHAT HAPPENS?

WE FIND THAT AN INCREASE OF 6 MINUTES BRINGS A PROFIT INCREASE OF ABOUT  $5 \times 0.1 = 0.5$  HUNDRED MILLION YEN.




THAT'S RIGHT. BUT, HOW MUCH DOES IT COST TO INCREASE THE AIRTIME OF THE COMMERCIAL BY 6 MINUTES?

THE FEE FOR THE INCREASE IS  $6 \times 0.1 = 0.6$  HUNDRED MILLION YEN.



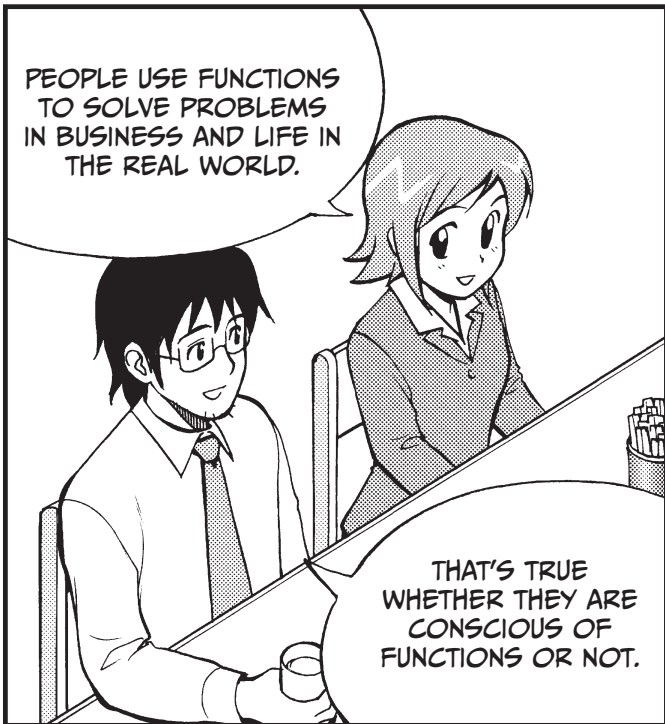
IF, INSTEAD, THE AIRTIME IS DECREASED BY 6 MINUTES, THE PROFIT DECREASES ABOUT 0.5 BILLION YEN. BUT SINCE YOU DON'T HAVE TO PAY THE FEE OF 0.6 HUNDRED MILLION YEN...



THE ANSWER IS...THE COMPANY  
DECIDED TO DECREASE THE  
COMMERCIAL TIME!

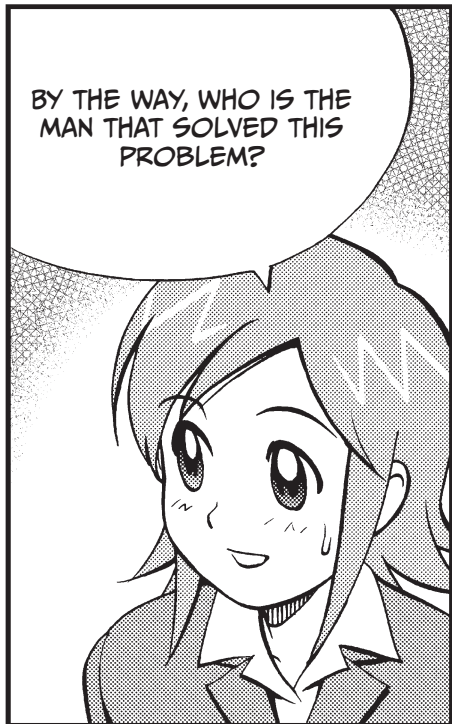


CORRECT!

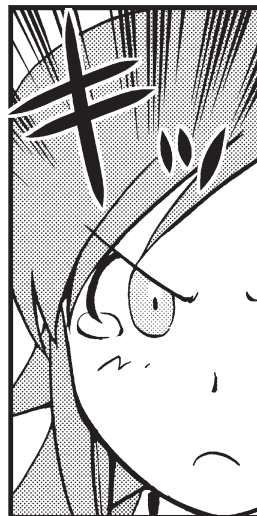
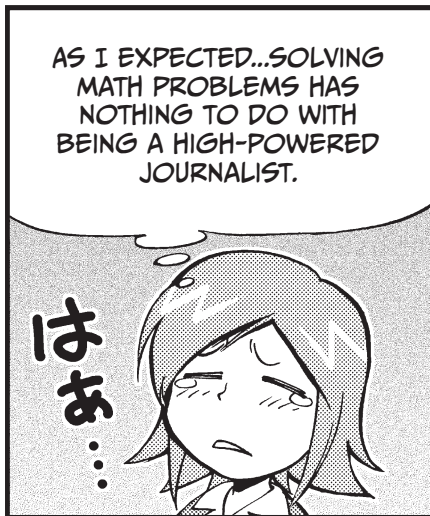


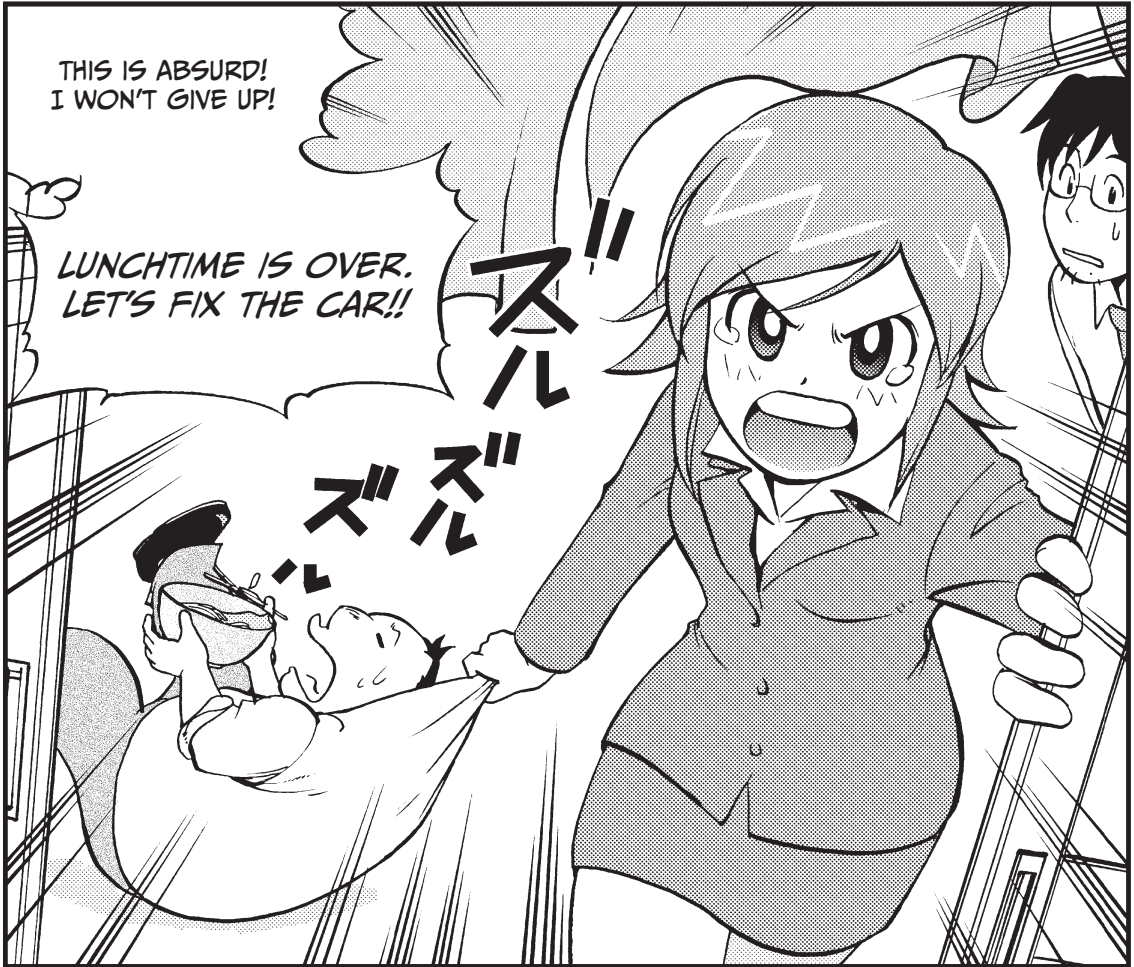
PEOPLE USE FUNCTIONS  
TO SOLVE PROBLEMS  
IN BUSINESS AND LIFE IN  
THE REAL WORLD.

THAT'S TRUE  
WHETHER THEY ARE  
CONSCIOUS OF  
FUNCTIONS OR NOT.



BY THE WAY, WHO IS THE  
MAN THAT SOLVED THIS  
PROBLEM?



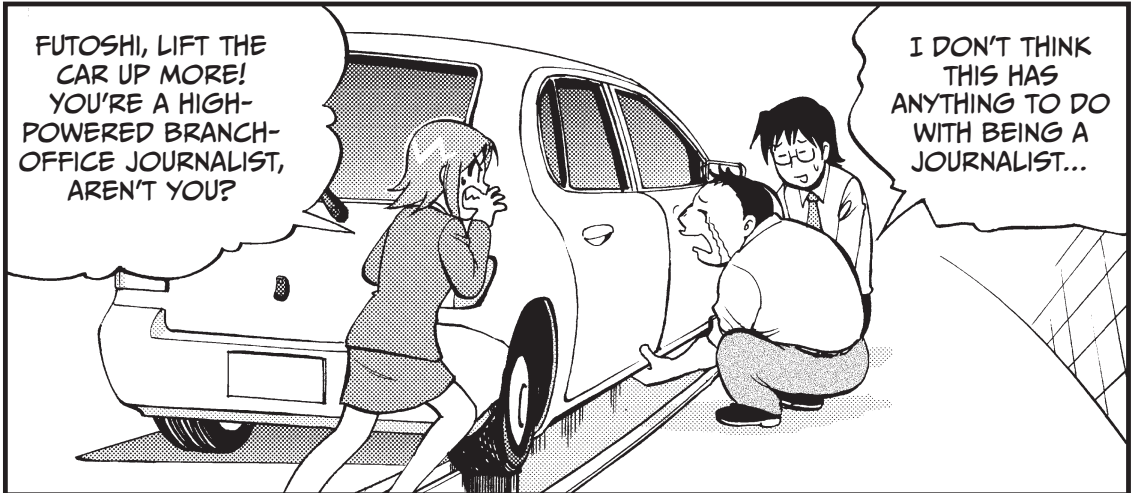


THIS IS ABSURD!  
I WON'T GIVE UP!

LUNCHTIME IS OVER.  
LET'S FIX THE CAR!!

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ズル



FUTOSHI, LIFT THE  
CAR UP MORE!  
YOU'RE A HIGH-  
POWERED BRANCH-  
OFFICE JOURNALIST,  
AREN'T YOU?

I DON'T THINK  
THIS HAS  
ANYTHING TO DO  
WITH BEING A  
JOURNALIST...

## CALCULATING THE DERIVATIVE

Let's find the imitating linear function  $g(x) = kx + l$  of function  $f(x)$  at  $x = a$ .  
We need to find slope  $k$ .

$$\textcircled{1} \quad g(x) = k(x - a) + f(a) \quad (g(x) \text{ coincides with } f(a) \text{ when } x = a.)$$

Now, let's calculate the relative error when  $x$  changes from  $x = a$  to  $x = a + \varepsilon$ .

$$\text{Relative error} = \frac{\text{Difference between } f \text{ and } g \text{ after } x \text{ has changed}}{\text{Change of } x \text{ from } x = a}$$

$$= \frac{f(a + \varepsilon) - g(a + \varepsilon)}{\varepsilon}$$

$$= \frac{f(a + \varepsilon) - (k\varepsilon + f(a))}{\varepsilon}$$

$$g(a + \varepsilon) = k(a + \varepsilon - a) + f(a) \\ = k\varepsilon + f(a)$$

$$= \frac{f(a + \varepsilon) - f(a)}{\varepsilon} - k \xrightarrow{\varepsilon \rightarrow 0} 0$$

When  $\varepsilon$  approaches 0,  
the relative error also  
approaches 0.

$$k = \lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

$$\frac{f(a + \varepsilon) - f(a)}{\varepsilon} \text{ approaches } k \\ \text{when } \varepsilon \rightarrow 0.$$

(The *lim* notation expresses the operation that obtains the value when  $\varepsilon$  approaches 0.)

Linear function  $\textcircled{1}$ , or  $g(x)$ , with this  $k$ , is an approximate function of  $f(x)$ .  
 $k$  is called the *differential coefficient* of  $f(x)$  at  $x = a$ .

$$\lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

Slope of the line tangent to  $y = f(x)$  at  
any point  $(a, f(a))$ .

We make symbol  $f'$  by attaching a prime to  $f$ .

$$f'(a) = \lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

$f'(a)$  is the slope of the line tangent to  
 $y = f(x)$  at  $x = a$ .

Letter  $a$  can be replaced with  $x$ .

Since  $f'$  can be seen as a function of  $x$ , it is called "the function  
derived from function  $f$ ," or the *derivative* of function  $f$ .

## CALCULATING THE DERIVATIVE OF A CONSTANT, LINEAR, OR QUADRATIC FUNCTION

1. Let's find the derivative of constant function  $f(x) = \alpha$ . The differential coefficient of  $f(x)$  at  $x = a$  is

$$\lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\alpha - \alpha}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} 0 = 0$$

Thus, the derivative of  $f(x)$  is  $f'(x) = 0$ . This makes sense, since our function is constant—the rate of change is 0.

**NOTE** The *differential coefficient* of  $f(x)$  at  $x = a$  is often simply called the derivative of  $f(x)$  at  $x = a$ , or just  $f'(a)$ .

2. Let's calculate the derivative of linear function  $f(x) = \alpha x + \beta$ . The derivative of  $f(x)$  at  $x = a$  is

$$\lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\alpha(a + \varepsilon) + \beta - (\alpha a + \beta)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \alpha = \alpha$$

Thus, the derivative of  $f(x)$  is  $f'(x) = \alpha$ , a constant value. This result should also be intuitive—linear functions have a constant rate of change by definition.

3. Let's find the derivative of  $f(x) = x^2$ , which appeared in the story. The differential coefficient of  $f(x)$  at  $x = a$  is

$$\lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{(a + \varepsilon)^2 - a^2}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{2a\varepsilon + \varepsilon^2}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} (2a + \varepsilon) = 2a$$

Thus, the differential coefficient of  $f(x)$  at  $x = a$  is  $2a$ , or  $f'(a) = 2a$ . Therefore, the derivative of  $f(x)$  is  $f'(x) = 2x$ .

## SUMMARY

- The calculation of a limit that appears in calculus is simply a formula calculating an error.
- A limit is used to obtain a derivative.
- The derivative is the slope of the tangent line at a given point.
- The derivative is nothing but the rate of change.

The derivative of  $f(x)$  at  $x = a$  is calculated by

$$\lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

$g(x) = f'(a)(x - a) + f(a)$  is then the *approximate linear function* of  $f(x)$ .  $f'(x)$ , which expresses the slope of the line tangent to  $f(x)$  at the point  $(x, f(x))$ , is called the *derivative* of  $f(x)$ , because it is derived from  $f(x)$ .

Other than  $f'(x)$ , the following symbols are also used to denote the derivative of  $y = f(x)$ .

$$y', \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx} f(x)$$

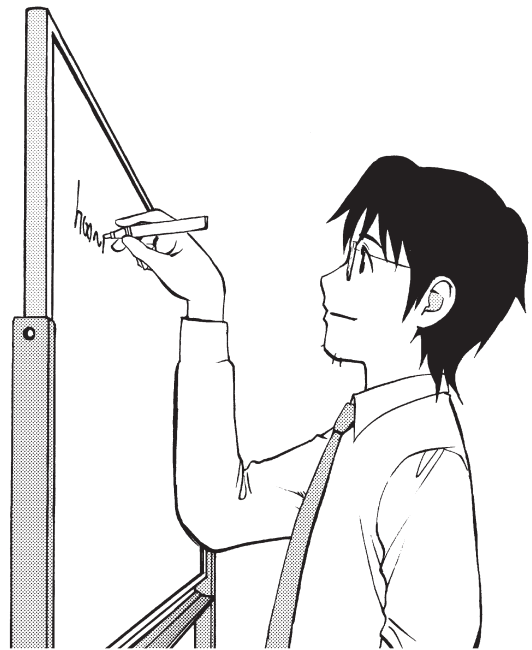
## EXERCISES

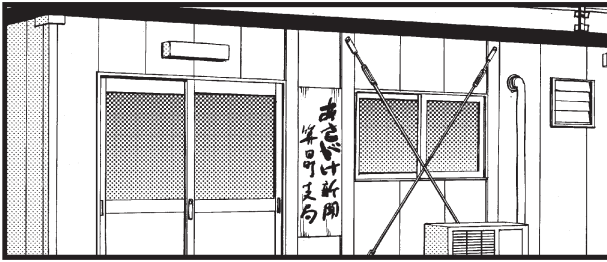
1. We have function  $f(x)$  and linear function  $g(x) = 8x + 10$ . It is known that the relative error of the two functions approaches 0 when  $x$  approaches 5.
  - A. Obtain  $f(5)$ .
  - B. Obtain  $f'(5)$ .
2. For  $f(x) = x^3$ , obtain its derivative  $f'(x)$ .



# 2

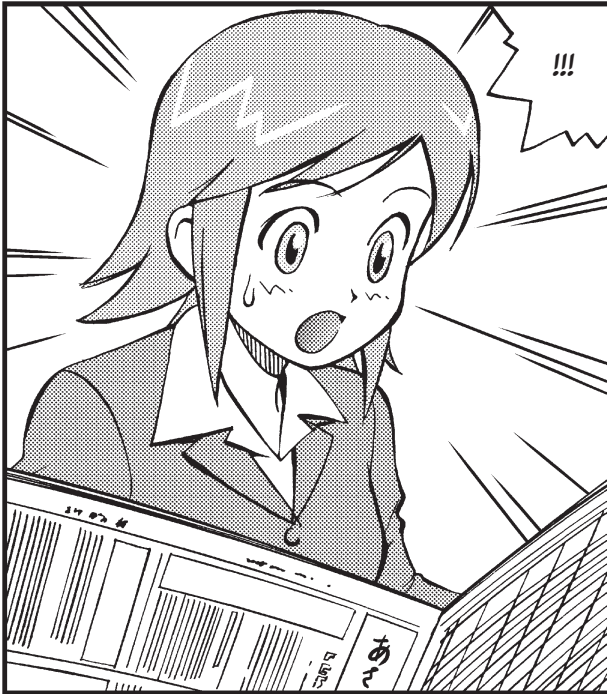
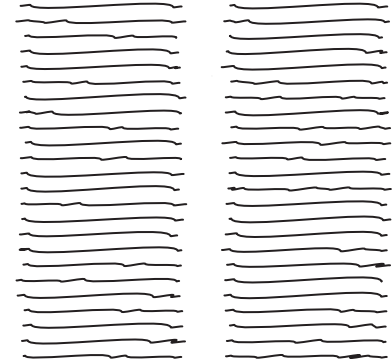
LET'S LEARN DIFFERENTIATION  
TECHNIQUES!



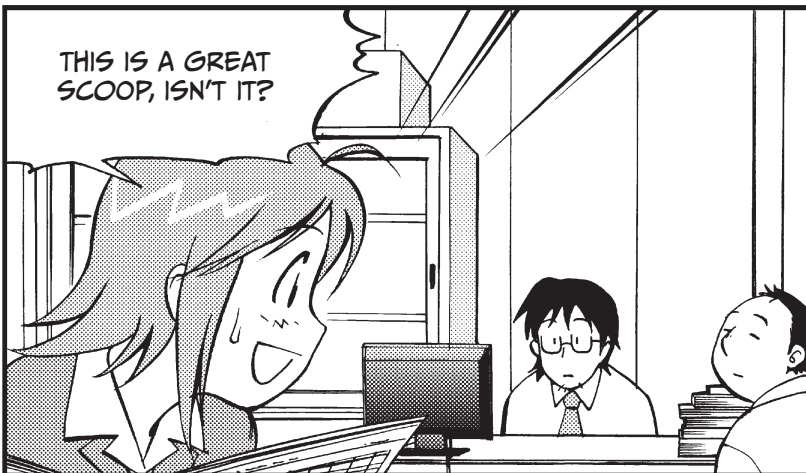


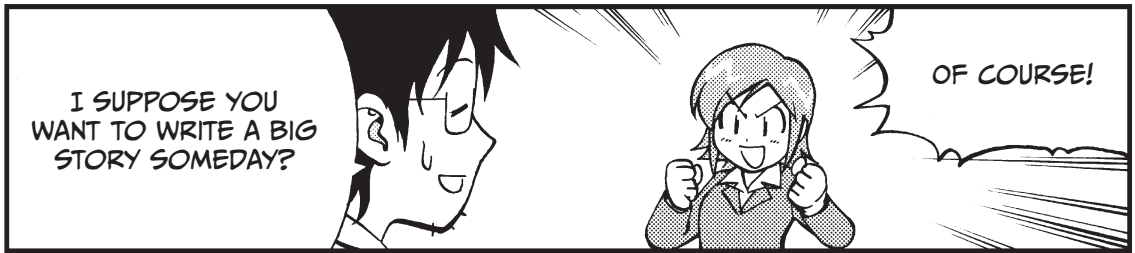
## Criminal Charges Brought Against Megatrox

### Construction Contract Violates Antitrust Laws



WOW! MEGATROX IS A HUGE COMPANY!



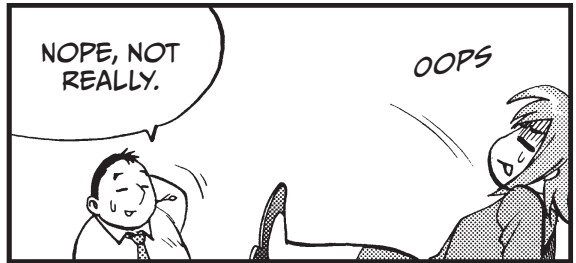


I SUPPOSE YOU WANT TO WRITE A BIG STORY SOMEDAY?

OF COURSE!

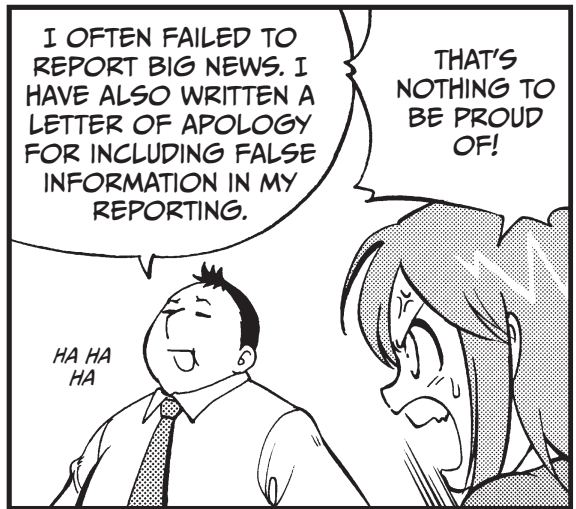


YOU TWO MUST HAVE GOT SOME REALLY EXCITING SCOOPS WHEN YOU WERE AT THE MAIN OFFICE. TELL ME!



NOPE, NOT REALLY.

OOPS



I OFTEN FAILED TO REPORT BIG NEWS. I HAVE ALSO WRITTEN A LETTER OF APOLOGY FOR INCLUDING FALSE INFORMATION IN MY REPORTING.

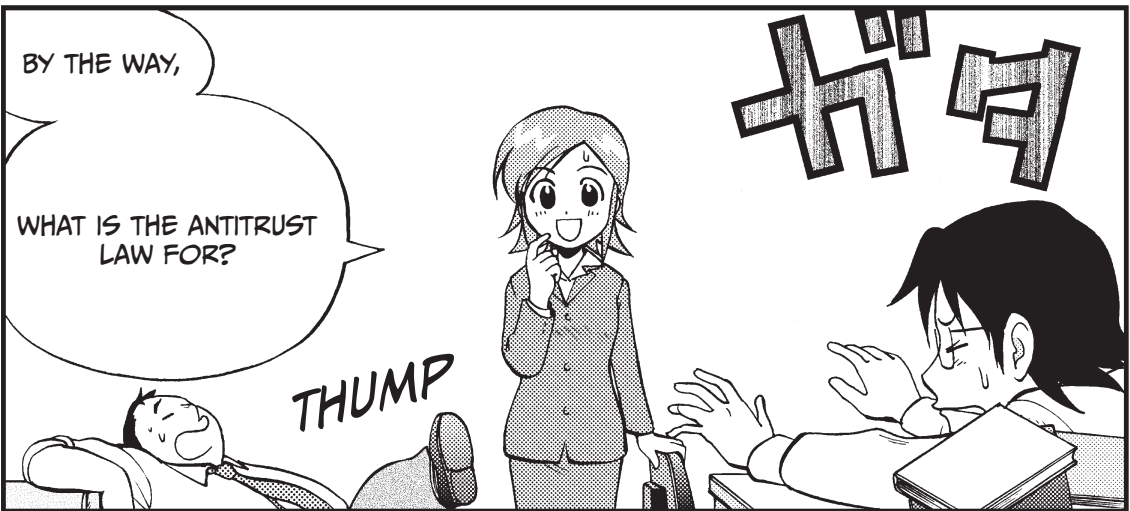
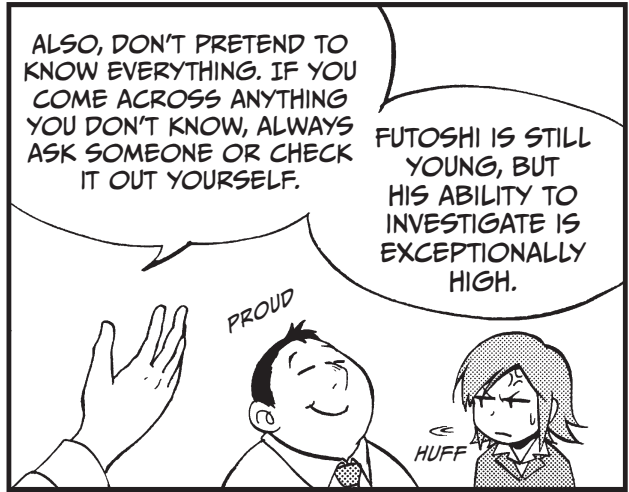
THAT'S NOTHING TO BE PROUD OF!

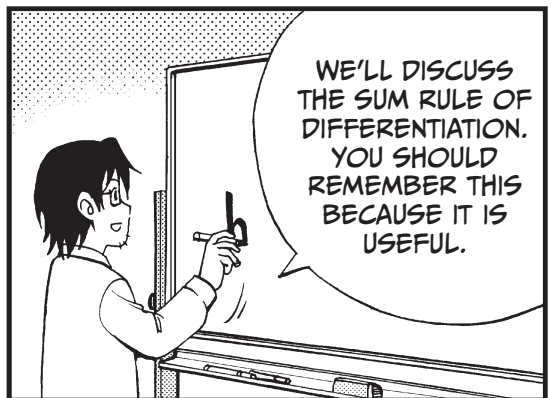
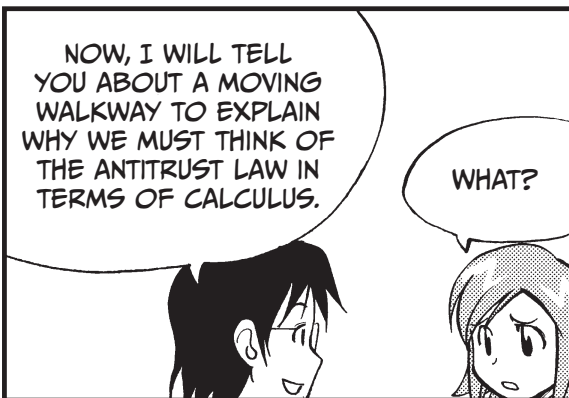
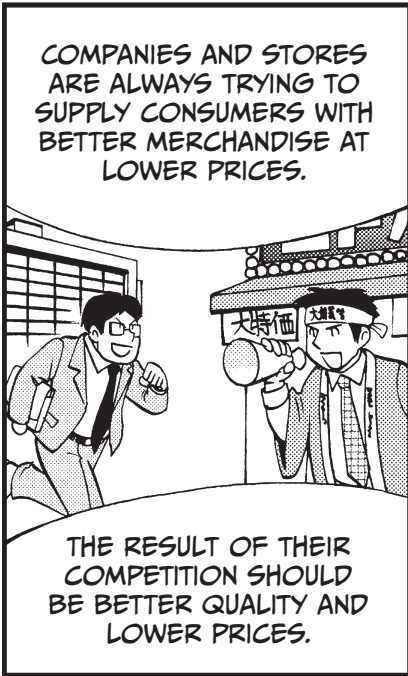
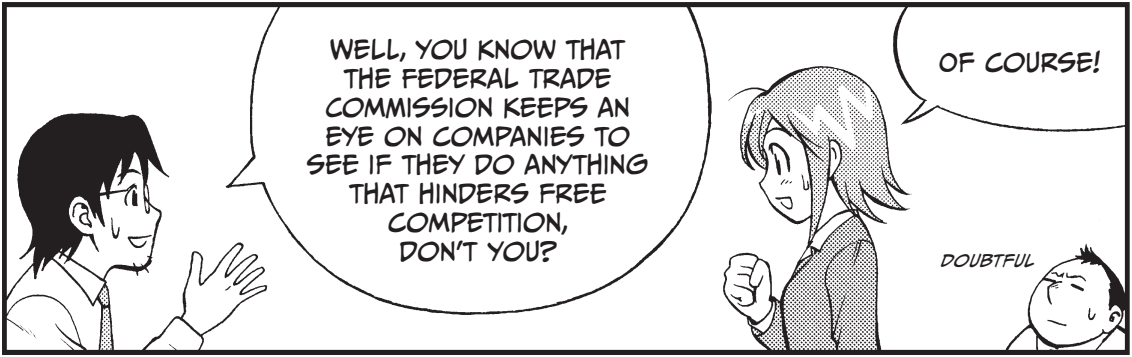
HA HA HA



CALM DOWN, NORIKO.

I UNDERSTAND THAT YOU HAVE HIGH EXPECTATIONS FOR NEWSPAPER JOURNALISM, BUT THE BASICS ARE MOST IMPORTANT.





# THE SUM RULE OF DIFFERENTIATION

## FORMULA 2-1: THE SUM RULE OF DIFFERENTIATION

For  $h(x) = f(x) + g(x)$   
 $h'(x) = f'(x) + g'(x)$

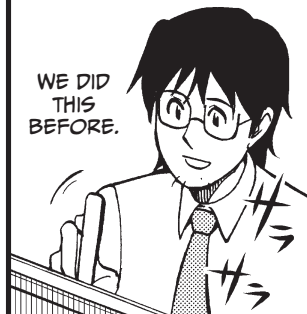
THAT IS, THE DERIVATIVE OF A FUNCTION IS EQUAL TO THE SUM OF THE DERIVATIVES OF THE FUNCTIONS THAT COMPOSE IT.



WHAT DOES THAT MEAN?

LET'S LOOK INTO THIS BY APPROXIMATING AROUND  $x = a$ .

WE DID THIS BEFORE.



$$f(x) \approx f'(a)(x-a) + f(a) \quad \textcircled{1}$$

APPROXIMATING

$$g(x) \approx g'(a)(x-a) + g(a) \quad \textcircled{2}$$

APPROXIMATING

SQUEAK

SQUEAK

SQUEAK

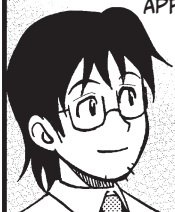


GIVEN THAT

$$h(x) \approx k(x-a) + l \quad \textcircled{3}$$

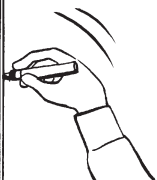
APPROXIMATING

WE WANT TO KNOW  $k$ .



SINCE  $h(x) = f(x) + g(x)$ ,  
SUBSTITUTE  $\textcircled{1}$  AND  $\textcircled{2}$  IN  
THIS EQUATION.

UH-HUH.



WE ALSO  
KNOW THAT...

$$h(x) \approx f'(a)(x-a) + f(a) + g'(a)(x-a) + g(a) \quad \textcircled{4}$$

SO IF WE REARRANGE  
THE TERMS OF  $\textcircled{4}$ ,  
EQUATION  $\textcircled{4}$  SAYS  
THE COEFFICIENT OF  
 $(x-a)$  WILL BE  $k$ .

LET'S SEE.

$$k = f'(a) + g'(a)!$$

RIGHT!

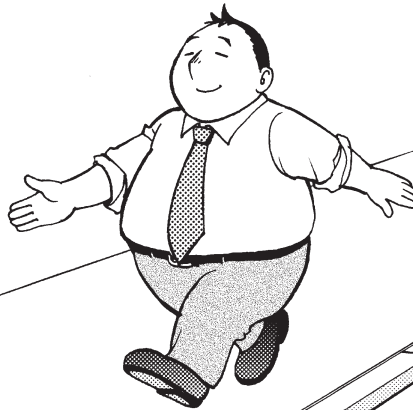
AND THE  
DIFFERENTIAL  
COEFFICIENT EQUALS  
THE DERIVATIVE. SO,

$$k = h'(a) = f'(a) + g'(a).$$

NOW, LET ME  
EXPLAIN ABOUT  
THE MOVING  
WALKWAY.

SUPPOSE FUTOSHI  
IS WALKING DOWN  
THE SIDEWALK.

I'D RATHER NOT  
THINK ABOUT IT,  
BUT I GUESS  
I WILL.



SUPPOSE THE DISTANCE HE WALKED IN  $x$  MINUTES FROM THE REFERENCE POINT 0 IS  $f(x)$  METERS.

$a$  MINUTES LATER, HE IS AT POINT A.

SUPPOSE  $x$  MINUTES LATER, HE IS AT POINT P.

THIS MEANS THAT HE TRAVELED FROM A TO P IN  $(x - a)$  MINUTES.

THAT'S RIGHT. BUT DOES IT MEAN ANYTHING?

SUPPOSE THIS TRAVEL TIME  $(x - a)$  IS EXTREMELY SHORT.

$$f(x) \approx f'(a)(x - a) + f(a)$$

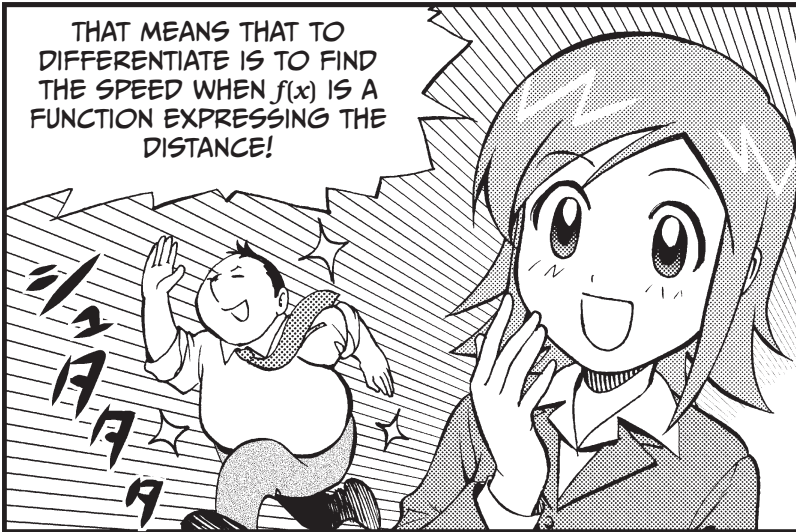
THIS CAN BE CHANGED INTO...

$$\frac{f(x) - f(a)}{x - a} \approx f'(a)$$

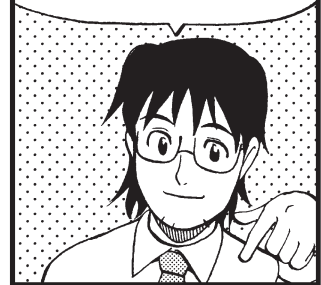
MR. SEKI, THE LEFT SIDE OF THIS EQUATION IS DISTANCE TRAVELED DIVIDED BY TRAVEL TIME. SO, IS THIS THE SPEED?

EXACTLY! SO,  $f'(a)$  REPRESENTS FUTOSHI'S SPEED WHEN HE PASSES POINT A.

THAT MEANS THAT TO DIFFERENTIATE IS TO FIND THE SPEED WHEN  $f(x)$  IS A FUNCTION EXPRESSING THE DISTANCE!



THAT'S RIGHT. SO, IF  $h(x) = f(x) + g(x)$ , THEN  $h'(x) = f'(x) + g'(x)$  MEANS THE FOLLOWING.

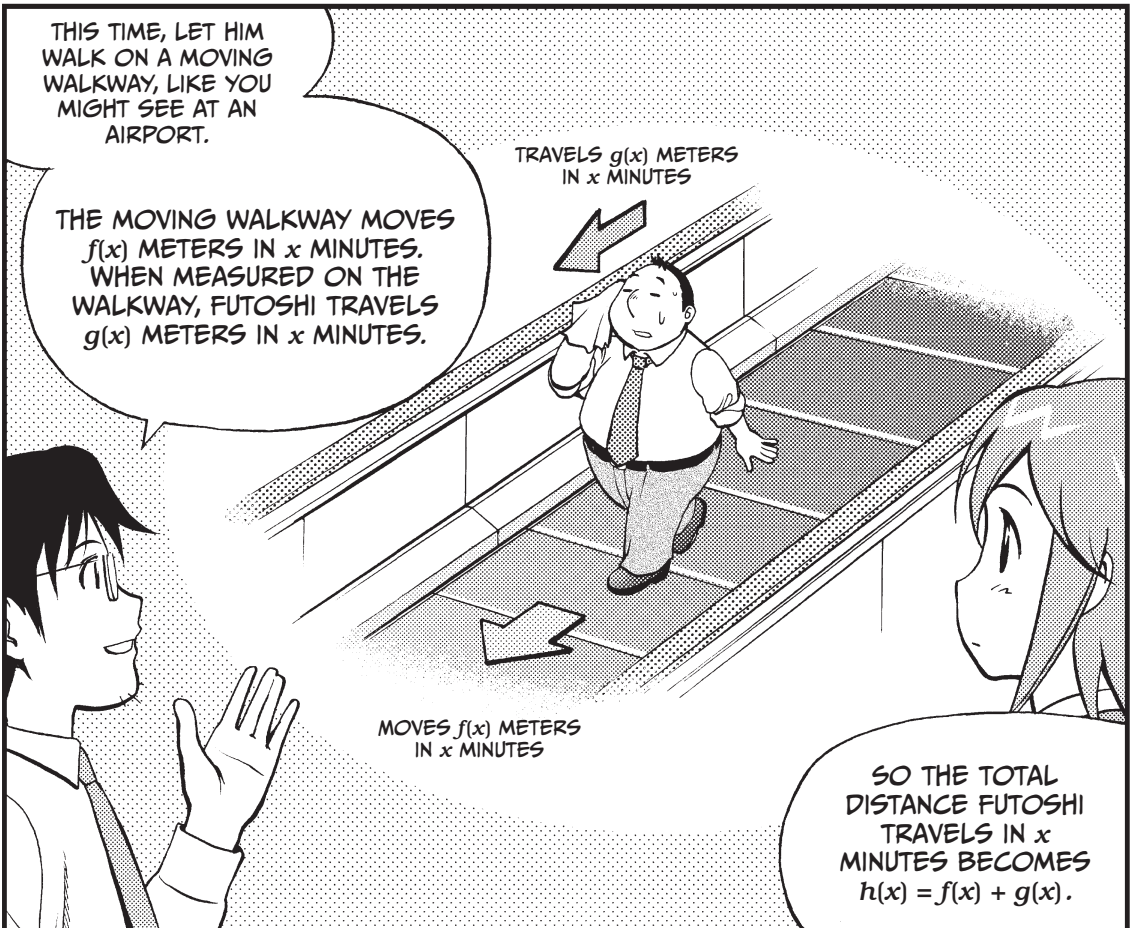


THIS TIME, LET HIM WALK ON A MOVING WALKWAY, LIKE YOU MIGHT SEE AT AN AIRPORT.

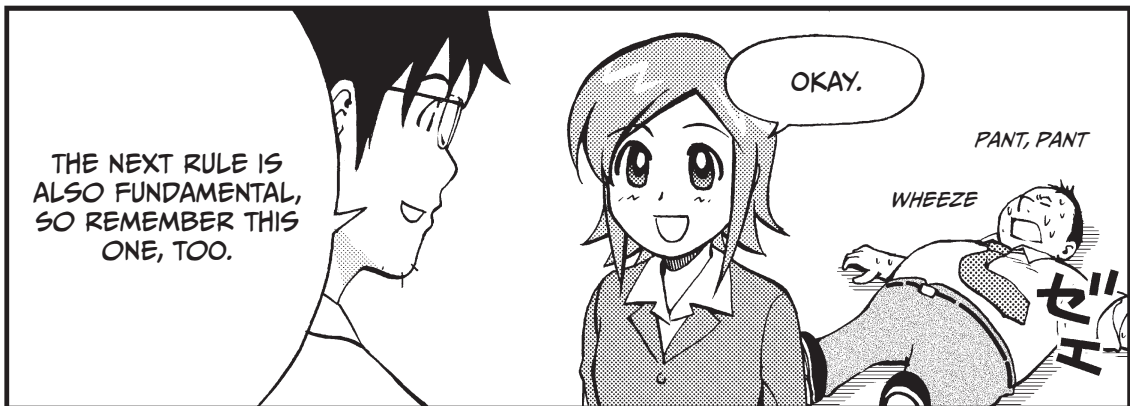
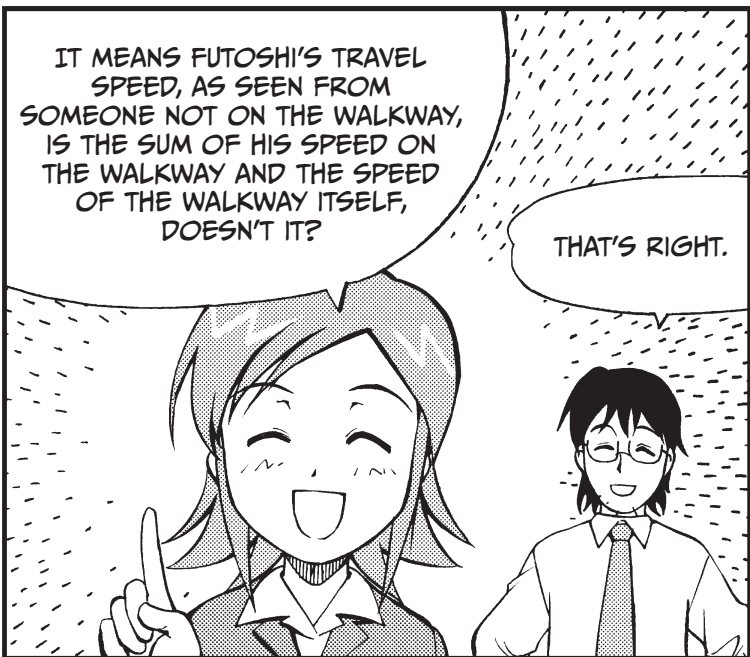
THE MOVING WALKWAY MOVES  $f(x)$  METERS IN  $x$  MINUTES. WHEN MEASURED ON THE WALKWAY, FUTOSHI TRAVELS  $g(x)$  METERS IN  $x$  MINUTES.

TRAVELS  $g(x)$  METERS IN  $x$  MINUTES

MOVES  $f(x)$  METERS IN  $x$  MINUTES



SO THE TOTAL DISTANCE FUTOSHI TRAVELS IN  $x$  MINUTES BECOMES  $h(x) = f(x) + g(x)$ .



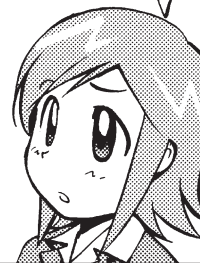
# THE PRODUCT RULE OF DIFFERENTIATION

**FORMULA 2-2:  
THE PRODUCT RULE OF DIFFERENTIATION**

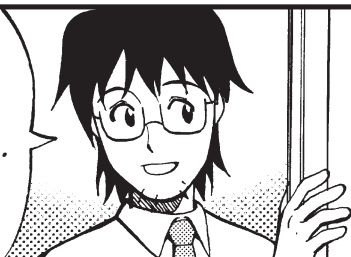
For  $h(x) = f(x)g(x)$   
 $h'(x) = f'(x)g(x) + f(x)g'(x)$

The derivative of a product is the sum of the products with only one function differentiated.

ONLY ONE FUNCTION?



YES. LET'S  
CONSIDER  $x = a$ .



$$f(x) \approx f'(a)(x-a) + f(a)$$

$$g(x) \approx g'(a)(x-a) + g(a)$$

$$h(x) = f(x)g(x) \approx k(x-a) + l$$

$$h(x) \approx \{f'(a)(x-a) + f(a)\} \times \{g'(a)(x-a) + g(a)\}$$

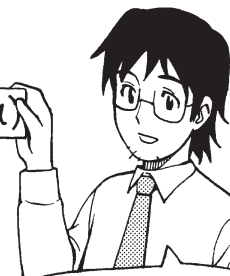
$$h(x) \approx f'(a)g'(a)(x-a)^2 + f(a)g'(a)(x-a) + f'(a)(x-a)g(a) + f(a)g(a)$$

$(x-a)$  IS A SMALL CHANGE. THAT MEANS  $(x-a)^2$  IS VERY, VERY SMALL. SINCE WE ARE APPROXIMATING, WE CAN THROW THAT TERM OUT.

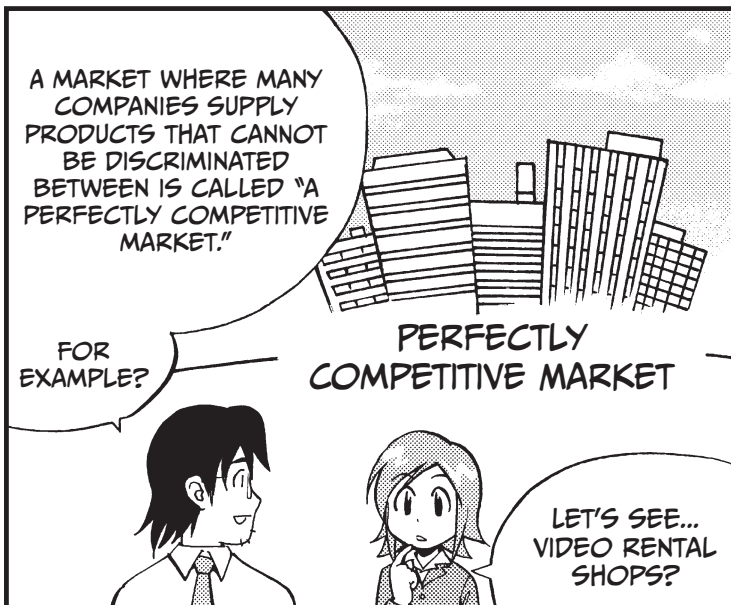
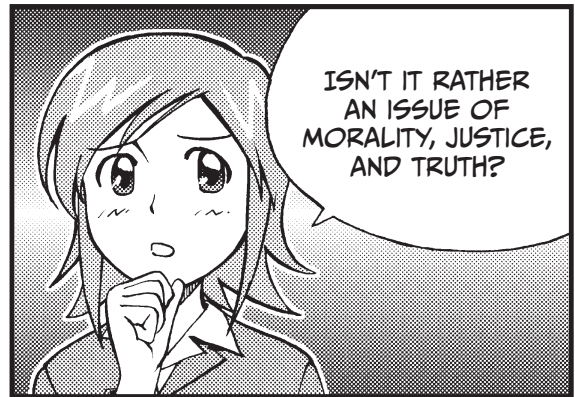
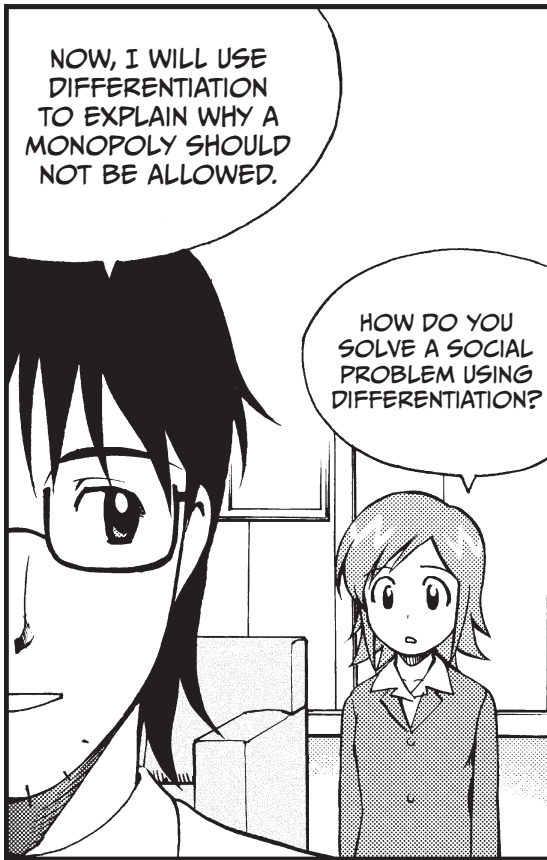
$$h(x) \approx \{f'(a)g(a) + f(a)g'(a)\}(x-a) + f(a)g(a)$$

$$k = f'(a)g(a) + f(a)g'(a)$$

$(x-a)$

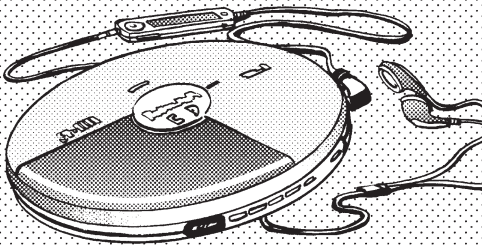


WE GET THIS.



\* IN REALITY, THERE ARE USUALLY BIG-NAME BRANDS FOR ANY COMMODITY. THERE ARE FAMOUS CHAIN SHOPS IN THE VIDEO RENTAL MARKET—NO MARKET CAN BE A PERFECTLY COMPETITIVE ONE, SO THIS IS A FICTITIOUS, IDEAL SITUATION.

SUPPOSE, FOR EXAMPLE, A COMPANY PRODUCING CD PLAYERS WHOSE MARKET PRICE IS ¥12,000 PER UNIT CONSIDERS WHETHER OR NOT IT WILL INCREASE PRODUCTION VOLUME.

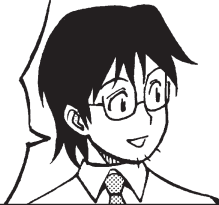


IF THE COST OF PRODUCING ONE MORE UNIT IS ¥10,000, THE COMPANY WILL SURELY INCREASE PRODUCTION, BECAUSE IT WILL MAKE MORE PROFIT.



SINCE MANY OTHER COMPANIES PRODUCE THE SAME KIND OF PRODUCT, THE COMPANY BELIEVES THAT ITS INCREASE IN PRODUCTION WILL CAUSE THE PRICE TO DECREASE.

SO THE COMPANY WILL CONSIDER MAKING ADDITIONAL UNITS. BUT THE COST OF MAKING ONE MORE UNIT CHANGES, AND THE COMPANY'S PRODUCTION EFFICIENCY WILL CHANGE. EVENTUALLY, THE COST OF MAKING ONE MORE UNIT WILL REACH THE MARKET PRICE OF ¥12,000. AT THAT POINT, AN INCREASE IN PRODUCTION WOULD NOT BE WORTH THE COST.



IN SHORT, THE MARKET STABILIZES WHEN THE MARKET PRICE OF THE UNIT EQUALS THE COST OF PRODUCING ANOTHER UNIT.

UH-HUH



ON THE OTHER HAND, THE STORY IS DIFFERENT IN A MONOPOLY MARKET, WHERE ONLY ONE COMPANY SUPPLIES A PARTICULAR PRODUCT. THEN JUST ONE COMPANY IS THE ENTIRE MARKET.

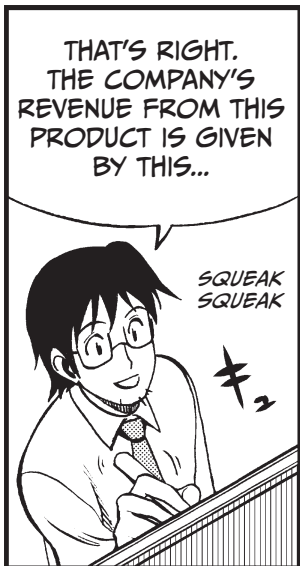
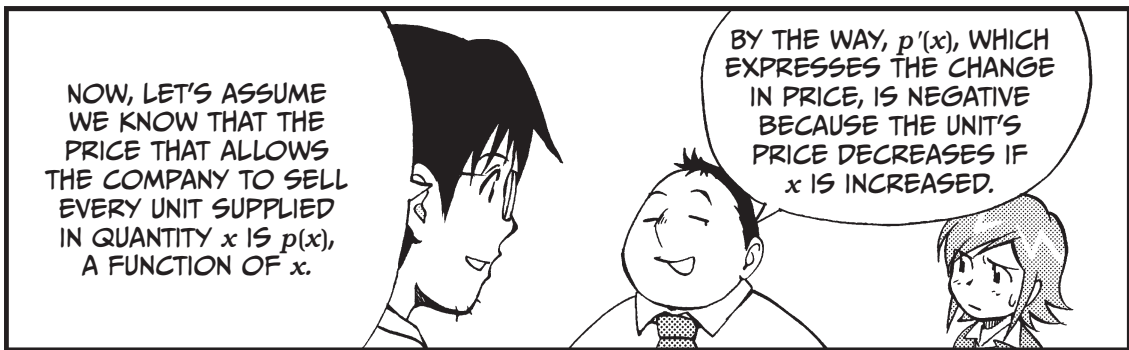
MONOPOLY



MARKET

WHEN YOU LOOK AT THE MARKET AS A WHOLE, AN INCREASE IN SUPPLY WILL CAUSE THE PRICE TO GO DOWN. THAT'S JUST SUPPLY AND DEMAND.





Revenue =  $R(x)$  = price  $\times$  quantity =  $p(x) \times x$

**FORMULA 2-3:  
THE COMPANY'S REVENUE**

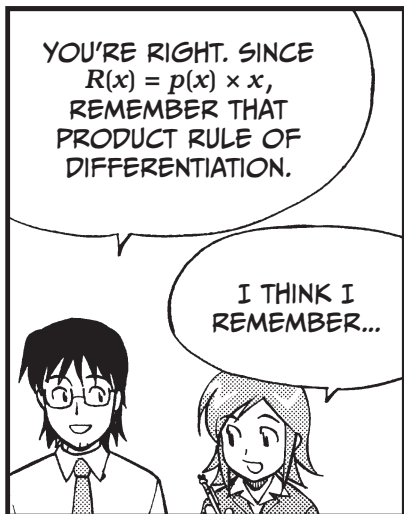
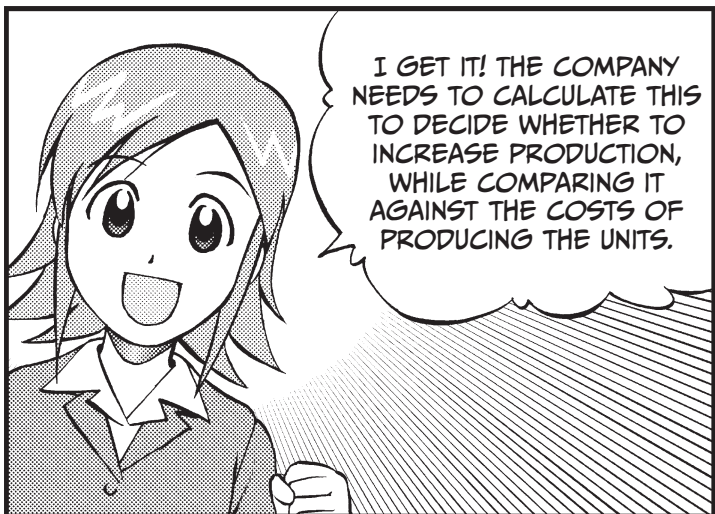
Since  $R(x) \approx R'(a)(x - a) + R(a)$   
we know that

$$R(x) - R(a) \approx R'(a)(x - a)$$

$\underbrace{\hspace{4em}}$   
CHANGE IN REVENUE

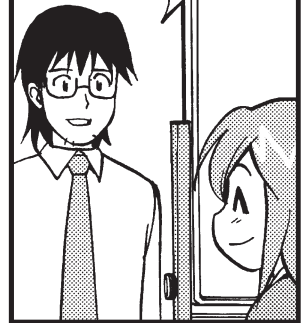
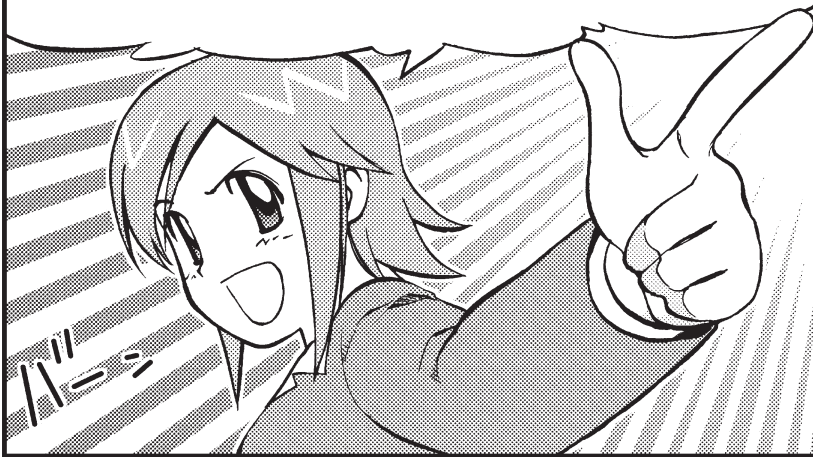
$\underbrace{\hspace{4em}}$   
CHANGE IN PRODUCTION VOLUME

THIS SHOWS US THAT THE ADDITIONAL REVENUE FROM AN INCREASE IN PRODUCTION IS  $R'(a)$  PER UNIT.



WE GET\*  $R'(a) = p'(a) \times a + p(a) \times 1$

RIGHT. PRODUCTION SHOULD BE STOPPED AT THE EXACT MOMENT IT BECOMES LESS THAN THE COST OF PRODUCTION INCREASE PER UNIT.

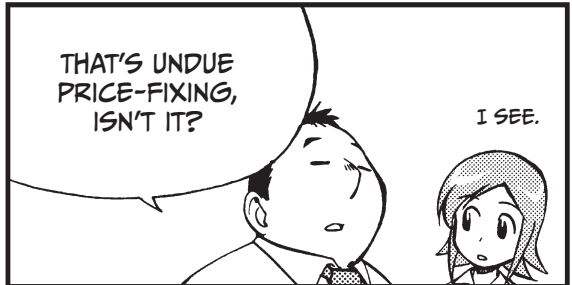


\* THE DERIVATIVE OF  $x$  IS 1 (SEE PAGE 40 FOR MORE ON DIFFERENTIATING LINEAR FUNCTIONS).

IN OTHER WORDS, PRODUCTION WILL BE STOPPED WHEN  $p'(a) \times a + p(a) = \text{COST OF PRODUCTION}$ . WE KNOW THAT THE FIRST TERM IS NEGATIVE, SO THE MARKET PRICE  $p(a)$  IS GREATER THAN THE COST.

THAT'S UNDUE PRICE-FIXING, ISN'T IT?

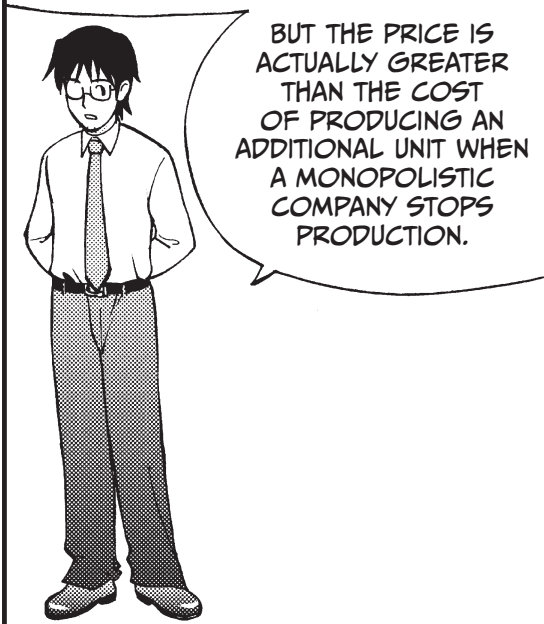
I SEE.



BUT THE PRICE IS ACTUALLY GREATER THAN THE COST OF PRODUCING AN ADDITIONAL UNIT WHEN A MONOPOLISTIC COMPANY STOPS PRODUCTION.

YOU ARE RIGHT, BUT YOU SHOULD TAKE A CLOSER LOOK. COMPANIES DO THIS NOT BECAUSE OF MALICIOUS MOTIVES BUT BASED ON A RATIONAL JUDGMENT.

LOOK AT THE EXPRESSION AGAIN.



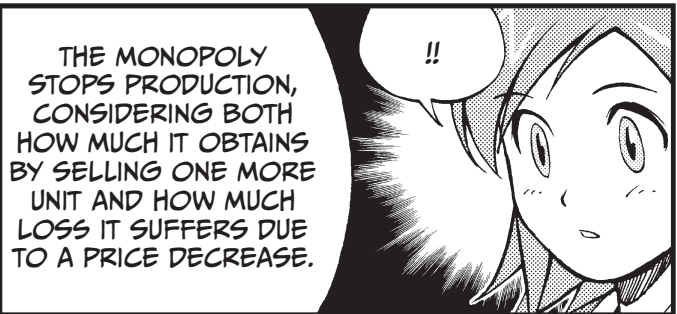
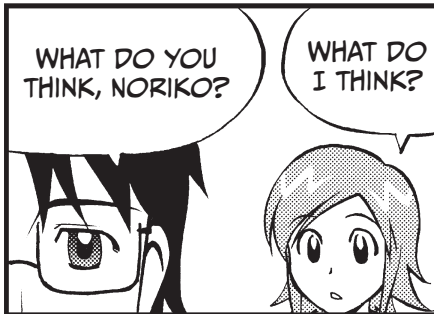
Sales increase (per unit) when production is increased a little more:

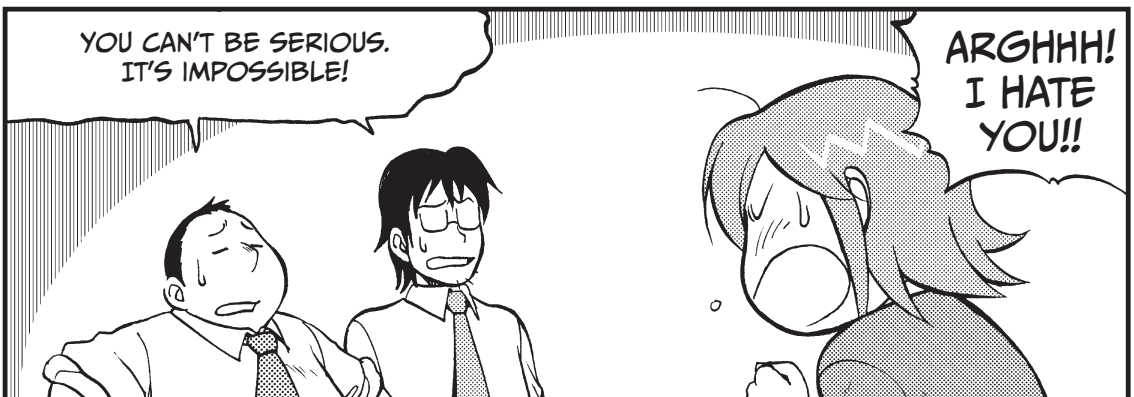
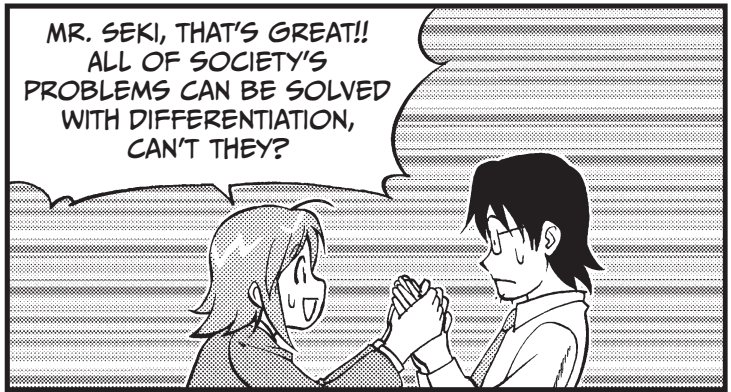
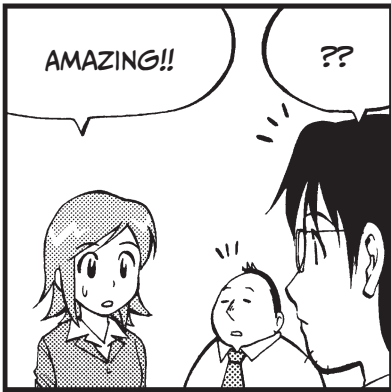
$$R'(a) = p'(a)a + p(a)$$

The two terms in the last expression mean the following:

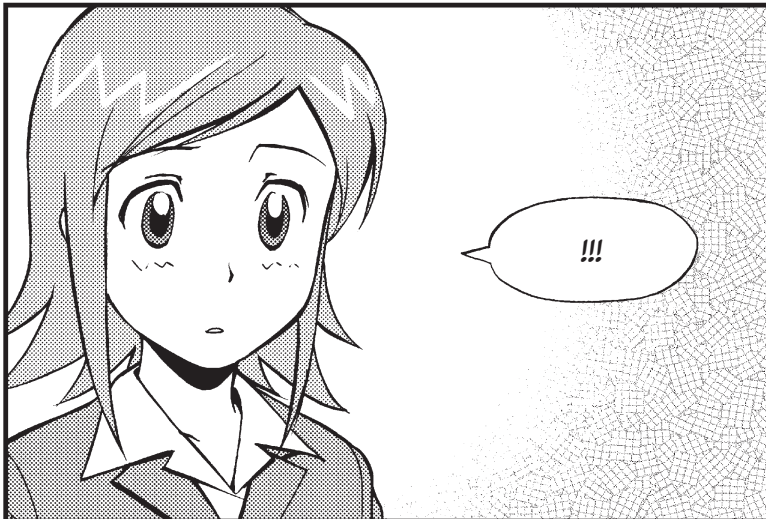
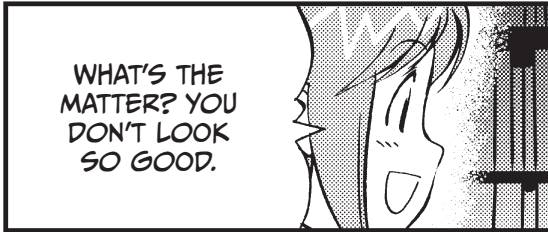
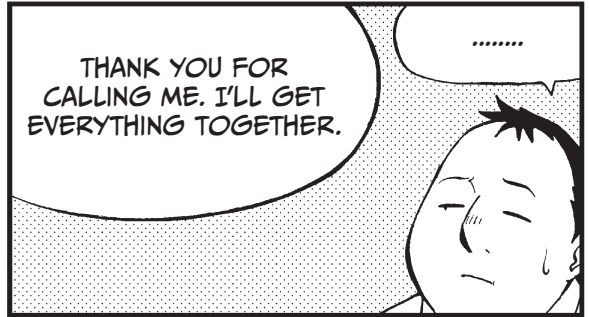
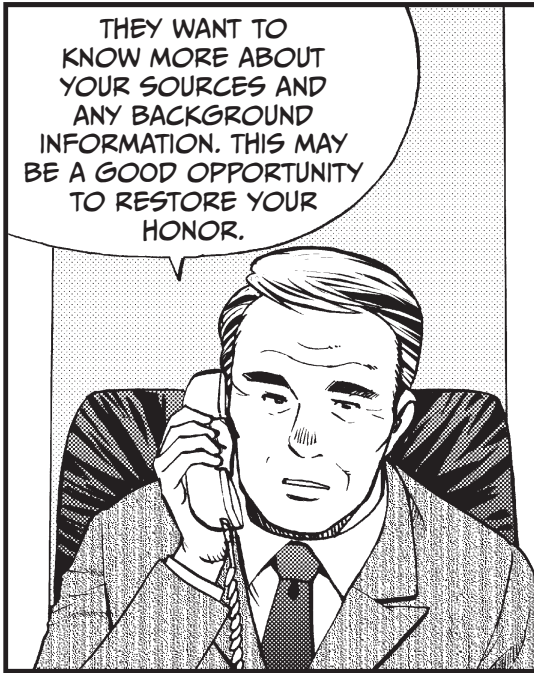
$p(a)$  represents the revenue from selling  $a$  units

$p'(a)a$  = Rate of price decrease  $\times$  Amount of production  
= A heavy loss due to price decrease influencing all units

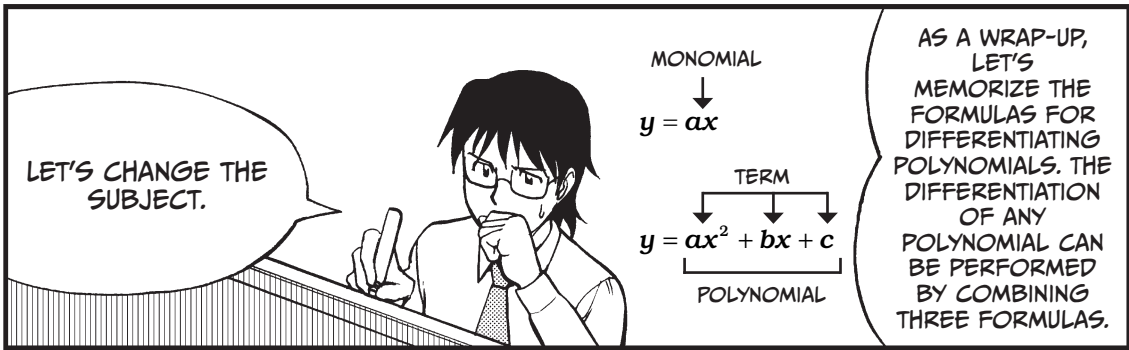








# DIFFERENTIATING POLYNOMIALS



LET'S CHANGE THE SUBJECT.

MONOMIAL  
 $y = ax$

TERM  
 $y = ax^2 + bx + c$   
 POLYNOMIAL

AS A WRAP-UP, LET'S MEMORIZE THE FORMULAS FOR DIFFERENTIATING POLYNOMIALS. THE DIFFERENTIATION OF ANY POLYNOMIAL CAN BE PERFORMED BY COMBINING THREE FORMULAS.

## FORMULA 2-4: THE DERIVATIVE OF AN $n$ TH-DEGREE FUNCTION

The derivative of  $h(x) = x^n$  is  $h'(x) = nx^{n-1}$

How do we get this general rule? We use the product rule of differentiation repeatedly.

For  $h(x) = x^2$ , since  $h(x) = x \times x$ ,  $h'(x) = x \times 1 + 1 \times x = 2x$

THIS RESULT IS USED

The formula is correct in this case.

For  $h(x) = x^3$ , since  $h(x) = x^2 \times x$ ,  $h'(x) = (x^2)' \times x + x^2 \times (x)' = (2x)x + x^2 \times 1 = 3x^2$

The formula is correct in this case, too.

For  $h(x) = x^4$ , since  $h(x) = x^3 \times x$ ,  $h'(x) = (x^3)' \times x + x^3 \times (x)' = 3x^2 \times x + x^3 \times 1 = 4x^3$

Again, the formula is correct. This continues forever. Any polynomial can be differentiated by combining the three formulas!

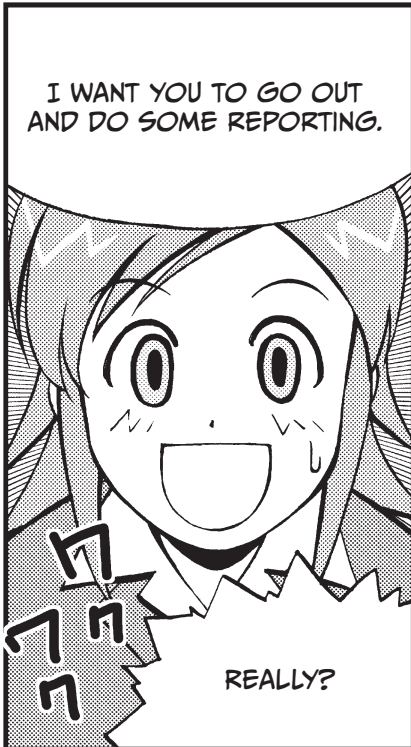
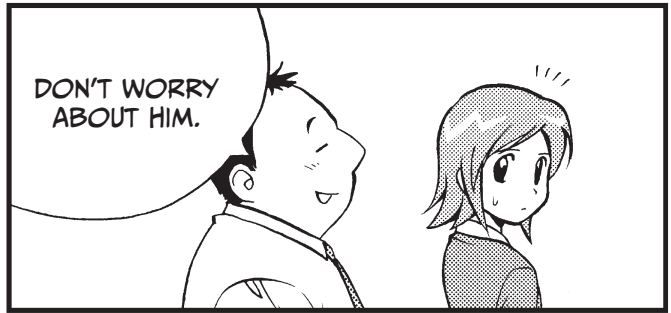
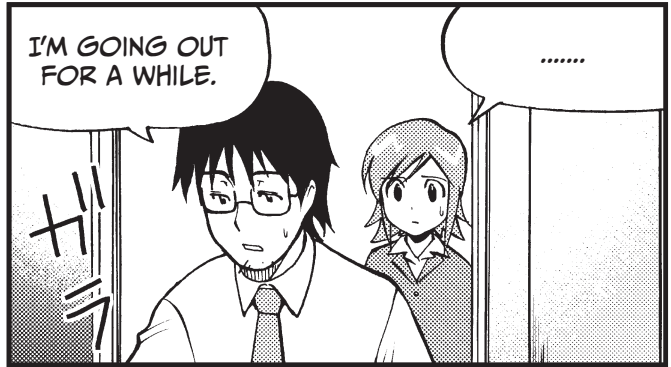
## FORMULA 2-5: THE DIFFERENTIATION FORMULAS OF SUM RULE, CONSTANT MULTIPLICATION, AND $x^n$

❶ Sum rule:  $\{f(x) + g(x)\}' = f'(x) + g'(x)$     ❸ Power rule ( $x^n$ ):  $\{x^n\}' = nx^{n-1}$

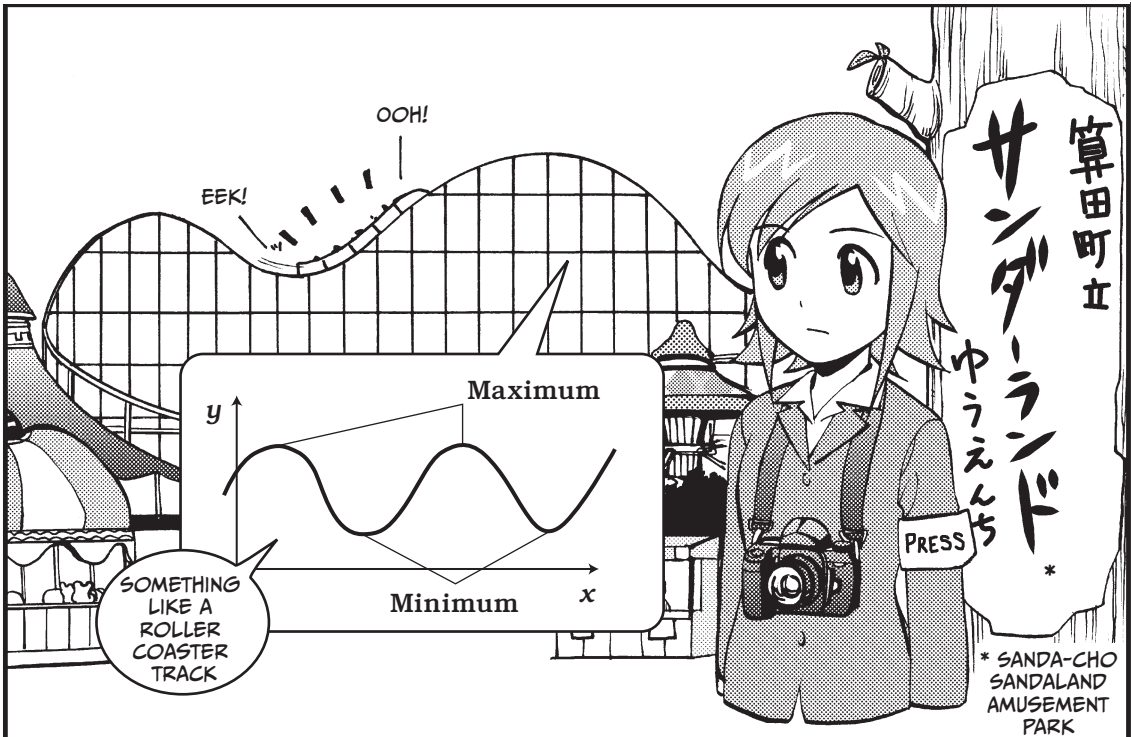
❷ Constant multiplication:  $\{af(x)\}' = af'(x)$

Let's see it in action! Differentiate  $h(x) = x^3 + 2x^2 + 5x + 3$

$$\begin{aligned}
 h'(x) &= \{x^3 + 2x^2 + 5x + 3\}' = \overbrace{(x^3)' + (2x^2)' + (5x)' + (3)'}^{\text{rule ❶}} \\
 &= \underbrace{(x^3)'}_{\text{rule ❷}} + 2 \underbrace{(x^2)'}_{\text{rule ❷}} + 5 \underbrace{(x)'}_{\text{rule ❷}} = 3x^2 + 2(2x) + 5 \times 1 = \underbrace{3x^2 + 4x + 5}_{\text{rule ❸}}
 \end{aligned}$$



# FINDING MAXIMA AND MINIMA



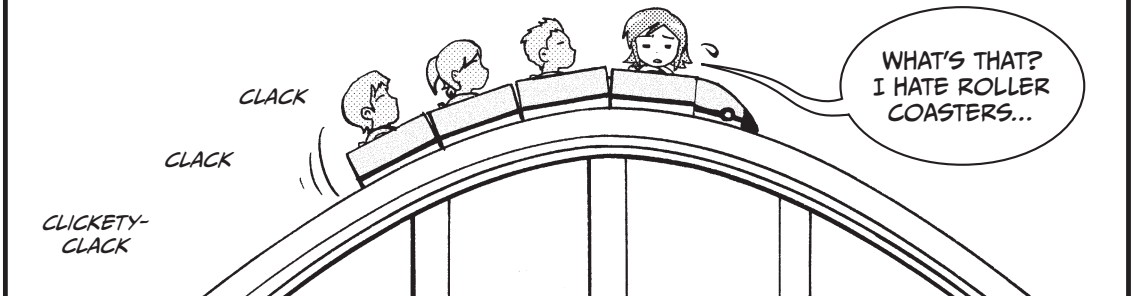
*Maxima* and *minima* are where a function changes from a decrease to an increase or vice versa. Thus they are important for examining the properties of a function.

Since a maximum or minimum is often the absolute maximum or minimum, respectively, it is an important point for obtaining an optimum solution.

## THEOREM 2-1: THE CONDITIONS FOR EXTREMA

If  $y = f(x)$  has a maximum or minimum at  $x = a$ , then  $f'(a) = 0$ .

This means that we can find maxima or minima by finding values of  $a$  that satisfy  $f'(a) = 0$ . These values are also called *extrema*.



Assume  $f'(a) > 0$ .

Since  $f(x) \approx f'(a)(x - a) + f(a)$  near  $x = a$ ,  $f'(a) > 0$  means that the approximate linear function is increasing at  $x = a$ . Thus, so is  $f(x)$ .

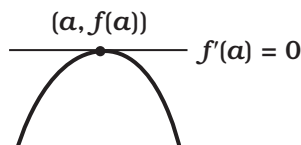
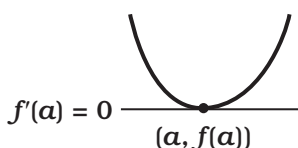
In other words, the roller coaster is ascending, and it is not at the top or at the bottom.

Similarly,  $y = f(x)$  is descending when  $f'(a) < 0$ , and it is not at the top or the bottom, either.

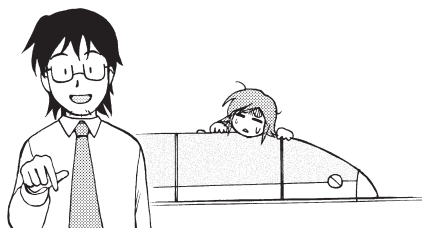


If  $y = f(x)$  is ascending or descending when  $f'(a) > 0$  or  $f'(a) < 0$ , respectively, we can only have  $f'(a) = 0$  at the top or bottom.

In fact, the approximate linear function  $y = f'(a)(x - a) + f(a) = 0 \times (x - a) + f(a)$  is a horizontal constant function when  $f'(a) = 0$ , which fits our understanding of maxima and minima.



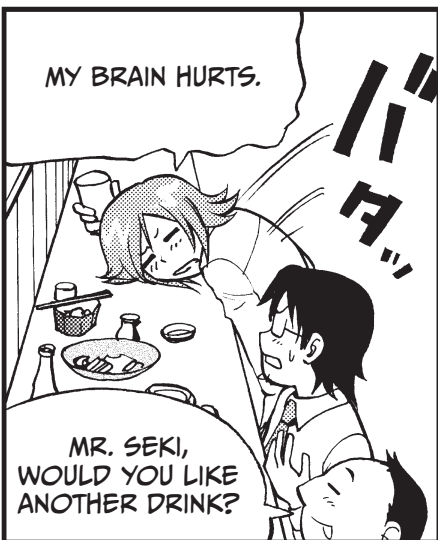
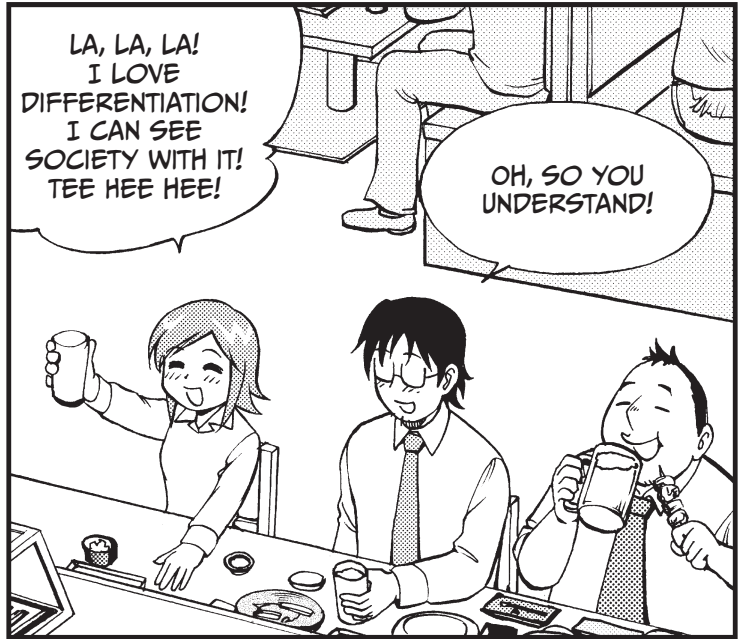
THIS DISCUSSION CAN BE SUMMARIZED INTO THE FOLLOWING THEOREM.

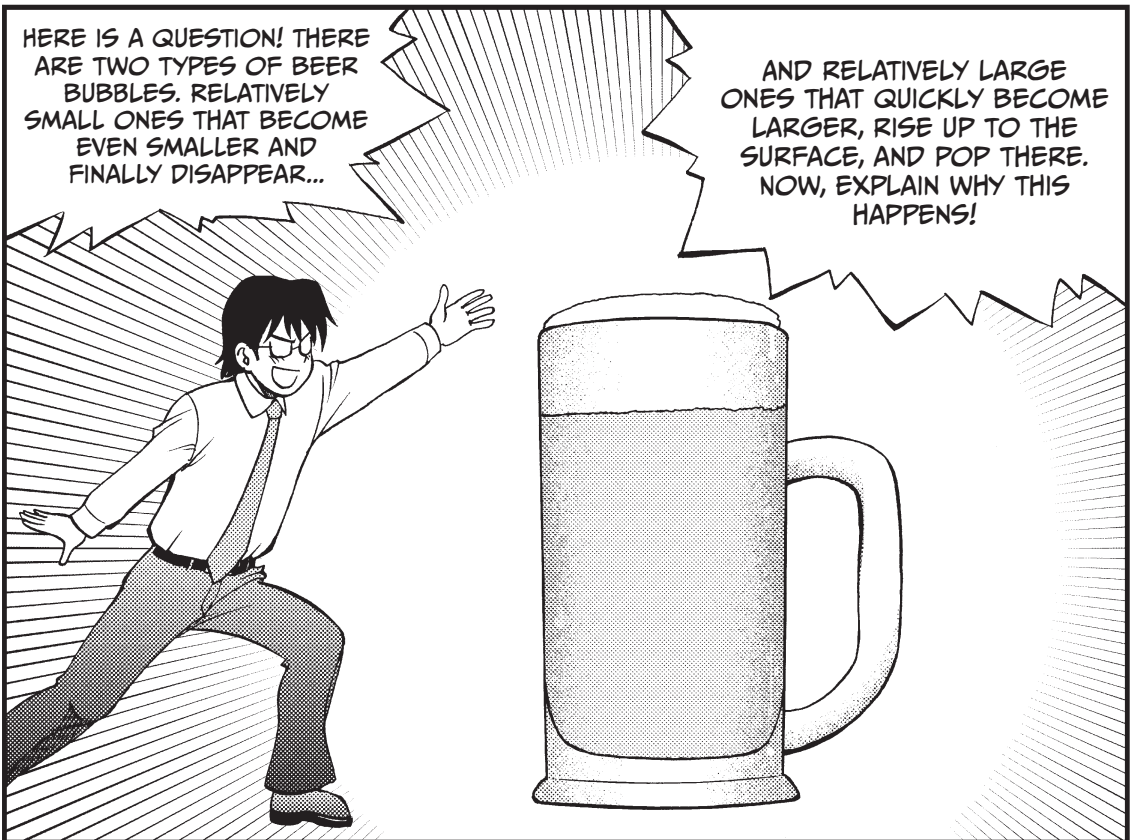


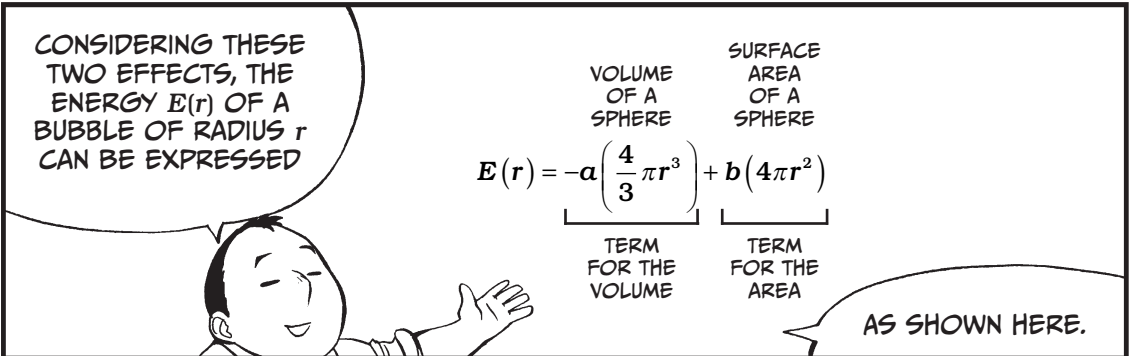
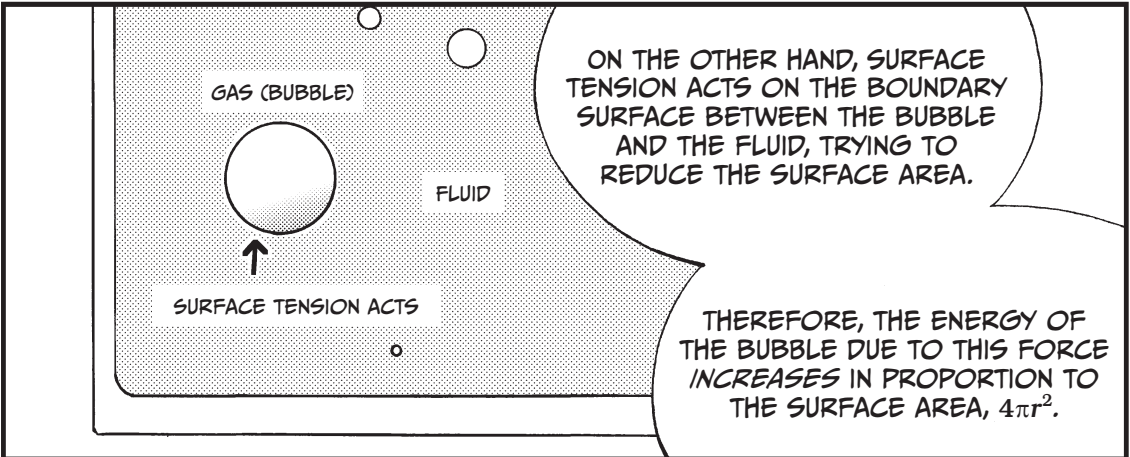
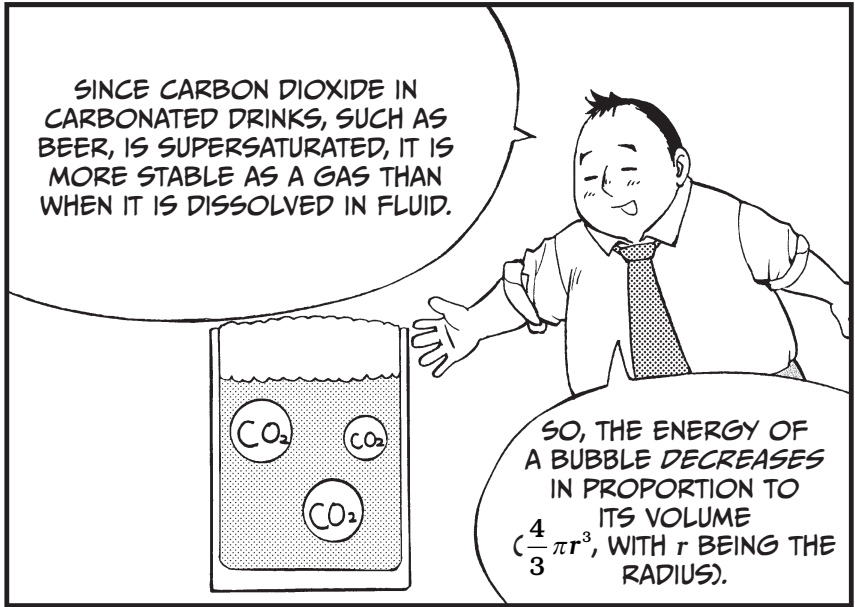
**THEOREM 2-2: THE CRITERIA FOR INCREASING AND DECREASING**

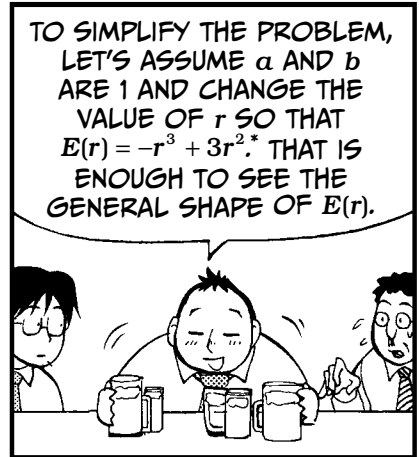
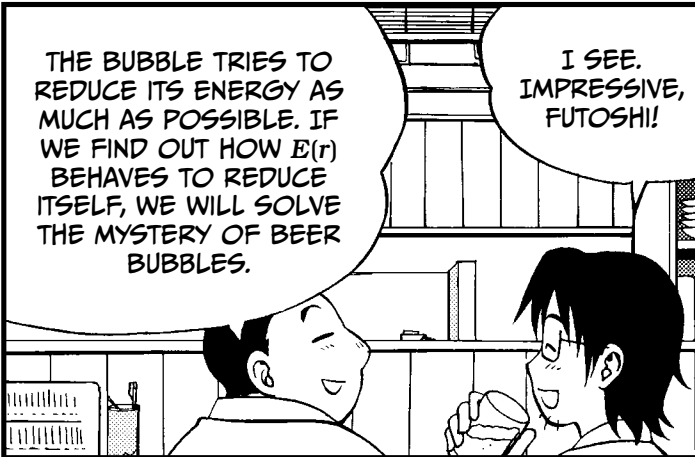
$y = f(x)$  is increasing around  $x = a$  when  $f'(a) > 0$ .

$y = f(x)$  is decreasing around  $x = a$  when  $f'(a) < 0$ .

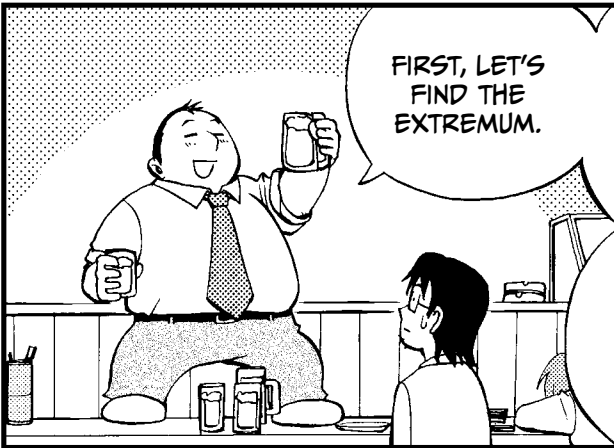








\* THIS IS CALLED NORMALIZING A VARIABLE. WE'VE SIMPLY MULTIPLIED EACH TERM BY  $3/(4\pi)$ .



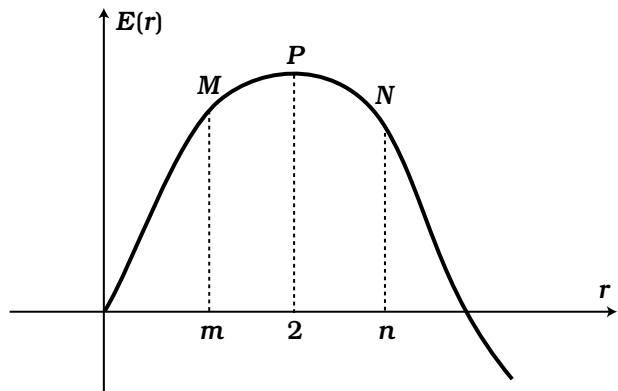
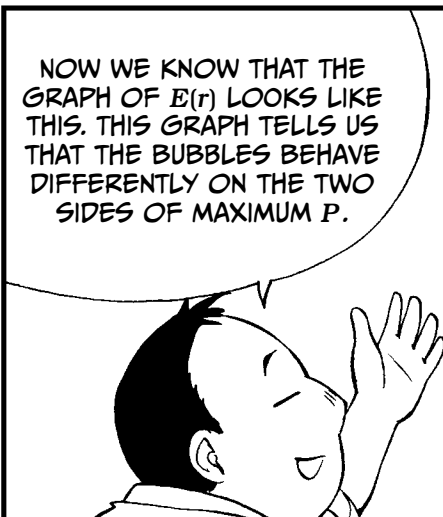
SINCE

$$E'(r) = (-r^3)' + (3r^2)'$$

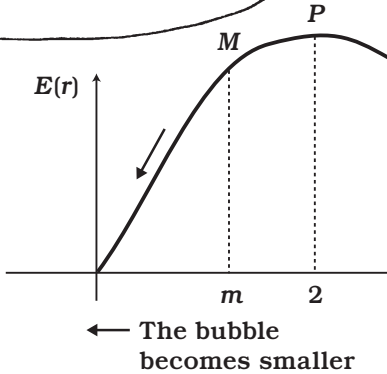
$$= -3r^2 + 6r$$

$$= -3r(r - 2)$$

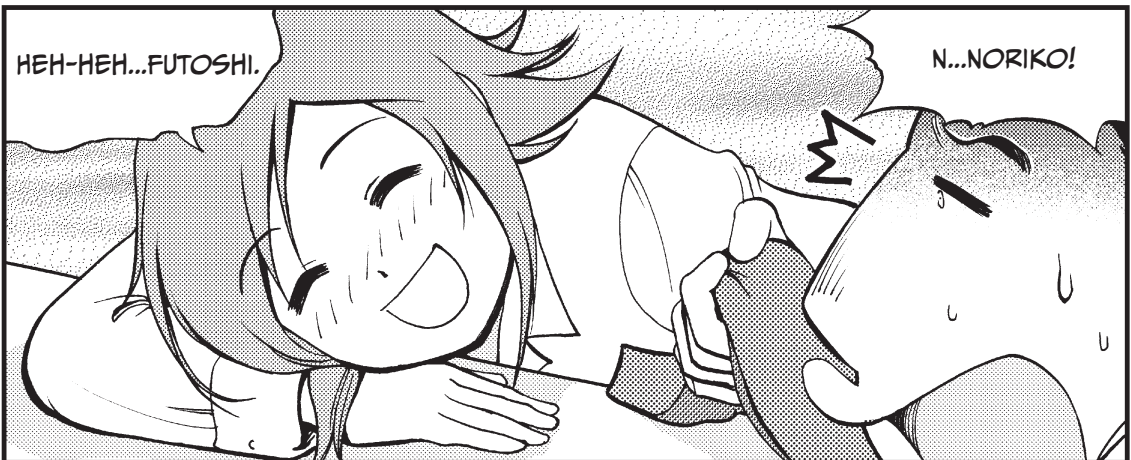
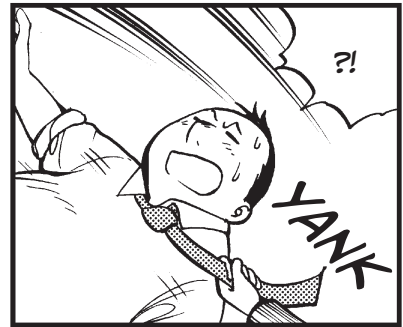
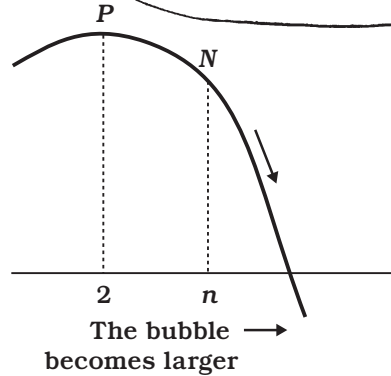
WHEN  $r = 2$ ,  $E'(r) = 0$ ,  
 FOR  $0 < r < 2$  ( $E'(r) > 0$ ), THE FUNCTION IS INCREASING, AND  
 FOR  $2 < r$ , THE FUNCTION IS DECREASING ( $E'(r) < 0$ ).  
 SO, WE FIND  $E(r)$  IS AT ITS MAXIMUM POINT  $P$  WHEN  $r = 2$ .

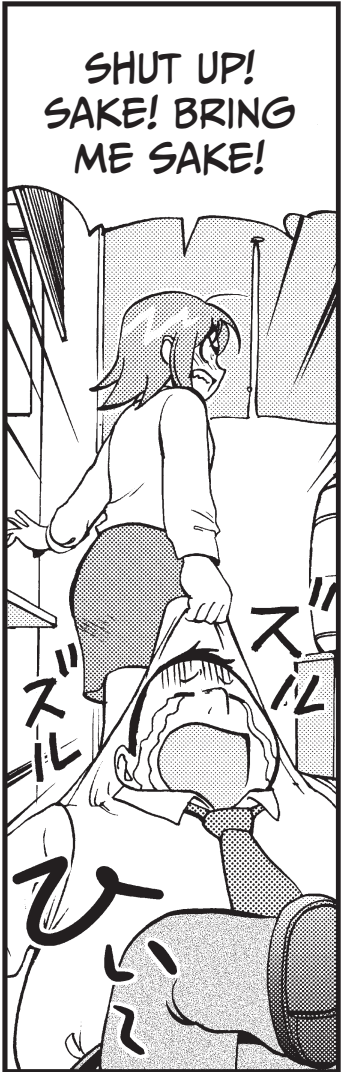


A BUBBLE THAT HAS THE RADIUS AND ENERGY OF POINT  $M$  SHOULD REDUCE ITS RADIUS UNTIL IT IS SMALLER THAN  $m$  TO MAKE ITS ENERGY  $E(r)$  SMALLER. THE BUBBLE WILL CONTINUE TO BECOME SMALLER UNTIL IT FINALLY DISAPPEARS.



ON THE OTHER HAND, A BUBBLE THAT HAS THE RADIUS AND ENERGY OF POINT  $N$  SHOULD INCREASE ITS RADIUS TO MAKE ITS ENERGY  $E(r)$  SMALLER. THE BUBBLE WILL CONTINUE TO GROW LARGER AND TO RISE UP INSIDE THE BEER.





## USING THE MEAN VALUE THEOREM

We saw before that the derivative is the coefficient of  $x$  in the approximate linear function that imitates function  $f(x)$  in the vicinity of  $x = a$ .

That is,

$$f(x) \approx f'(a)(x - a) + f(a) \quad (\text{when } x \text{ is very close to } a)$$

But the linear function only “pretends to be” or “imitates”  $f(x)$ , and for  $b$ , which is near  $a$ , we generally have

$$\textcircled{1} \quad f(b) \neq f'(a)(b - a) + f(a)$$

So, this is not exactly an equation.



FOR THOSE WHO CANNOT STAND FOR THIS, WE HAVE THE FOLLOWING THEOREM.

### THEOREM 2-3: THE MEAN VALUE THEOREM

For  $a$ ,  $b$  ( $a < b$ ), and  $c$ , which satisfy  $a < c < b$ , there exists a number  $c$  that satisfies

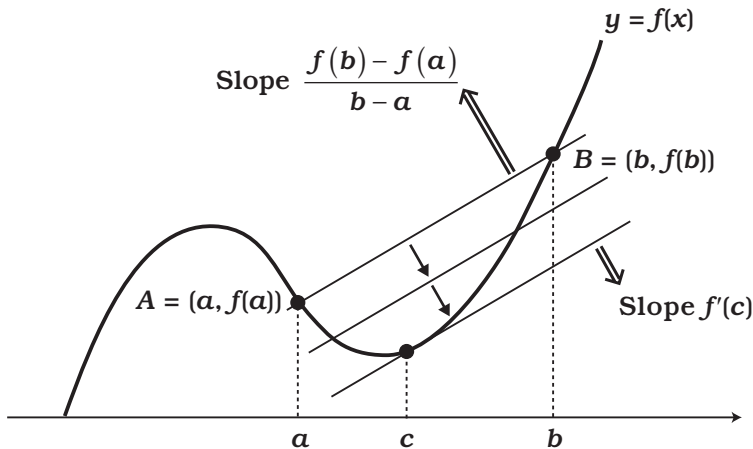
$$f(b) = f'(c)(b - a) + f(a)$$

In other words, we can make expression  $\textcircled{1}$  hold with an equal sign not with  $f'(a)$  but with  $f'(c)$ , where  $c$  is a value existing somewhere between  $a$  and  $b$ .\*



\* That is, there must be a value for  $x$  between  $a$  and  $b$  (which we'll call  $c$ ) that has a tangent line matching the slope of a line connecting points A and B.

Let's draw a line through point  $A = (a, f(a))$  and point  $B = (b, f(b))$  to form line segment  $AB$ .



We know the slope is simply  $\Delta y / \Delta x$ :

$$\textcircled{2} \quad \text{Slope of } AB = \frac{f(b) - f(a)}{b - a}$$

Now, move line  $AB$  parallel to its initial state as shown in the figure.

The line eventually comes to a point beyond which it separates from the graph. Denote this point by  $(c, f(c))$ .

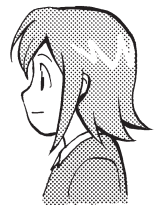
At this moment, the line is a tangent line, and its slope is  $f'(c)$ .

Since the line has been moved parallel to the initial state, this slope has not been changed from slope  $\textcircled{2}$ .

THEREFORE, WE KNOW

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

MULTIPLY BOTH SIDES BY THE DENOMINATOR AND TRANSPOSE TO GET  $f(b) = f'(c)(b - a) + f(a)$



## USING THE QUOTIENT RULE OF DIFFERENTIATION

Let's find the formula for the derivative of  $h(x) = \frac{g(x)}{f(x)}$

First, we find the derivative of function  $p(x) = \frac{1}{f(x)}$ , which is the reciprocal of  $f(x)$ .

If we know this, we'll be able to apply the product rule to  $h(x)$ .

Using simple algebra, we see that  $f(x)p(x) = 1$  always holds.

$$1 = f(x)p(x) \approx \{f'(a)(x-a) + f(a)\}\{p'(a)(x-a) + p(a)\}$$

Since these two are equal, their derivatives must be equal as well.

$$0 = p(x)f'(x) + p'(x)f(x)$$

Thus, we have  $p'(x) = -\frac{p(x)f'(x)}{f(x)}$ .

Since  $p(a) = \frac{1}{f(a)}$ , substituting this for  $p(a)$  in the numerator gives

$$p'(a) = \frac{-f'(a)}{f(a)^2}.$$

For  $h(x) = \frac{g(x)}{f(x)}$  in general, we consider  $h(x) = g(x) \square \frac{1}{f(x)} = g(x)p(x)$

and use the product rule and the above formula.

$$\begin{aligned} h'(x) &= g'(x)p(x) + g(x)p'(x) = g'(x)\frac{1}{f(x)} - g(x)\frac{f'(x)}{f(x)^2} \\ &= \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2} \end{aligned}$$

Therefore, we obtain the following formula.

FORMULA 2-6: THE QUOTIENT RULE OF DIFFERENTIATION

$$h'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2}$$

## CALCULATING DERIVATIVES OF COMPOSITE FUNCTIONS

Let's obtain the formula for the derivative of  $h(x) = g(f(x))$ .

Near  $x = a$ ,

$$f(x) - f(a) \approx f'(a)(x - a)$$

And near  $y = b$ ,

$$g(y) - g(b) \approx g'(b)(y - b)$$

We now substitute  $b = f(a)$  and  $y = f(x)$  in the last expression.

Near  $x = a$ ,

$$g(f(x)) - g(f(a)) \approx g'(f(a))(f(x) - f(a))$$

Replace  $f(x) - f(a)$  in the right side with the right side of the first expression.

$$g(f(x)) - g(f(a)) \approx g'(f(a))f'(a)(x - a)$$

Since  $g(f(x)) = h(x)$ , the coefficient of  $(x - a)$  in this expression gives us  $h'(a) = g'(f(a))f'(a)$ .

We thus obtain the following formula.

**FORMULA 2-7: THE DERIVATIVES OF COMPOSITE FUNCTIONS**

$$h'(x) = g'(f(x))f'(x)$$

## CALCULATING DERIVATIVES OF INVERSE FUNCTIONS

Let's use the above formula to find the formula for the derivative of  $x = g(y)$ , the inverse function of  $y = f(x)$ .

Since  $x = g(f(x))$  for any  $x$ , differentiating both sides of this expression gives  $1 = g'(f(x))f'(x)$ .

Thus,  $1 = g'(y)f'(x)$ , and we obtain the following formula.

**FORMULA 2-8: THE DERIVATIVES OF INVERSE FUNCTIONS**

$$g'(y) = \frac{1}{f'(x)}$$

FORMULAS OF DIFFERENTIATION

	FORMULA	KEY POINT
Constant multiplication	$\{\alpha f(x)\}' = \alpha f'(x)$	The multiplicative constant can be factored out.
$x^n$ (Power)	$(x^n)' = nx^{n-1}$	The exponent becomes the coefficient, reducing the degree by 1.
Sum	$\{f(x) + g(x)\}' = f'(x) + g'(x)$	The derivative of a sum is the sum of the derivatives.
Product	$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$	The sum of the products with each function differentiated in turn.
Quotient	$\left\{\frac{g(x)}{f(x)}\right\}' = \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2}$	The denominator is squared. The numerator is the difference between the products with only one function differentiated.
Composite functions	$\{g(f(x))\}' = g'(f(x))f'(x)$	The product of the derivative of the outer and that of the inner.
Inverse functions	$g'(y) = \frac{1}{f'(x)}$	The derivative of an inverse function is the reciprocal of the original.

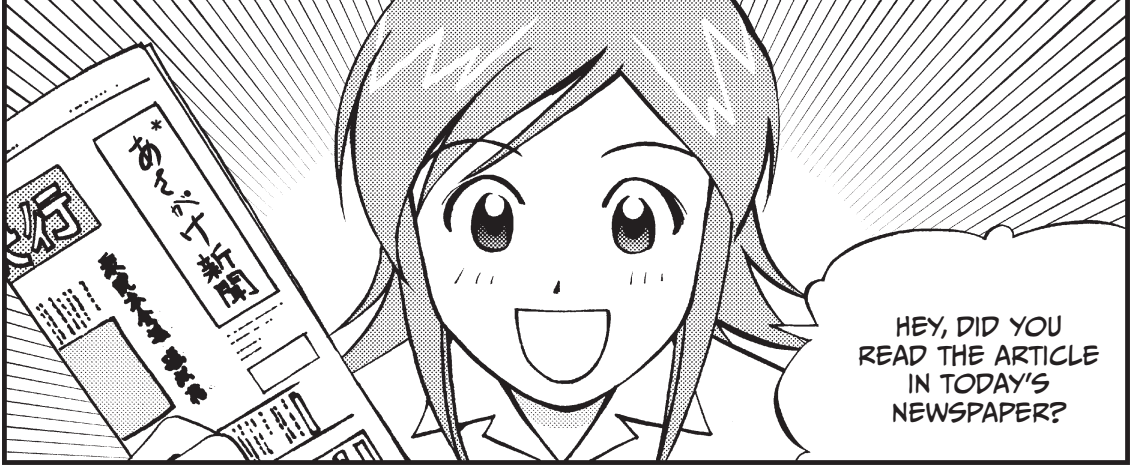
EXERCISES

- For natural number  $n$ , find the derivative  $f'(x)$  of  $f(x) = \frac{1}{x^n}$ .
- Calculate the extrema of  $f(x) = x^3 - 12x$ .
- Find the derivative  $f'(x)$  of  $f(x) = (1 - x)^3$ .
- Calculate the maximum value of  $g(x) = x^2(1 - x)^3$  in the interval  $0 \leq x \leq 1$ .

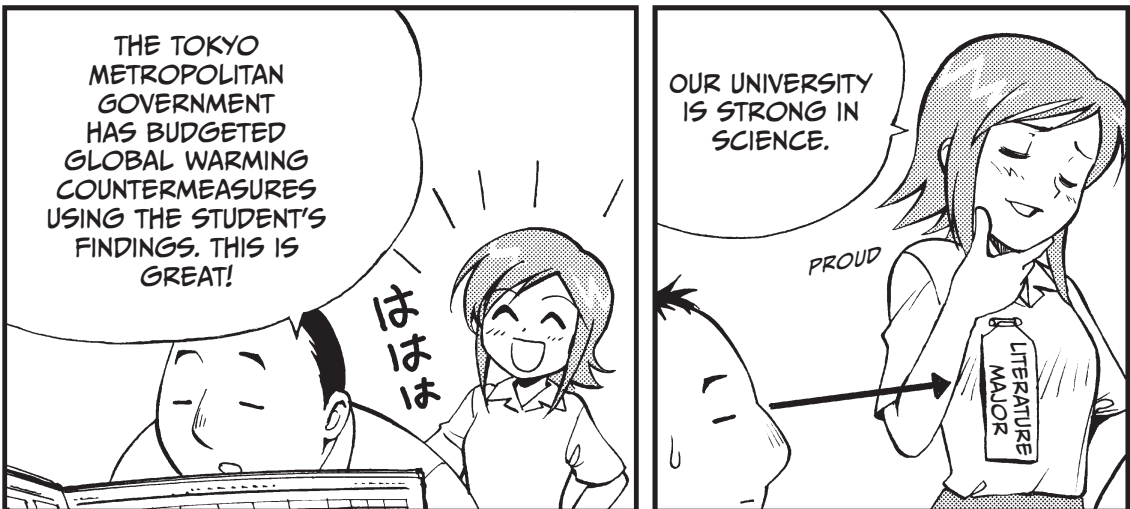
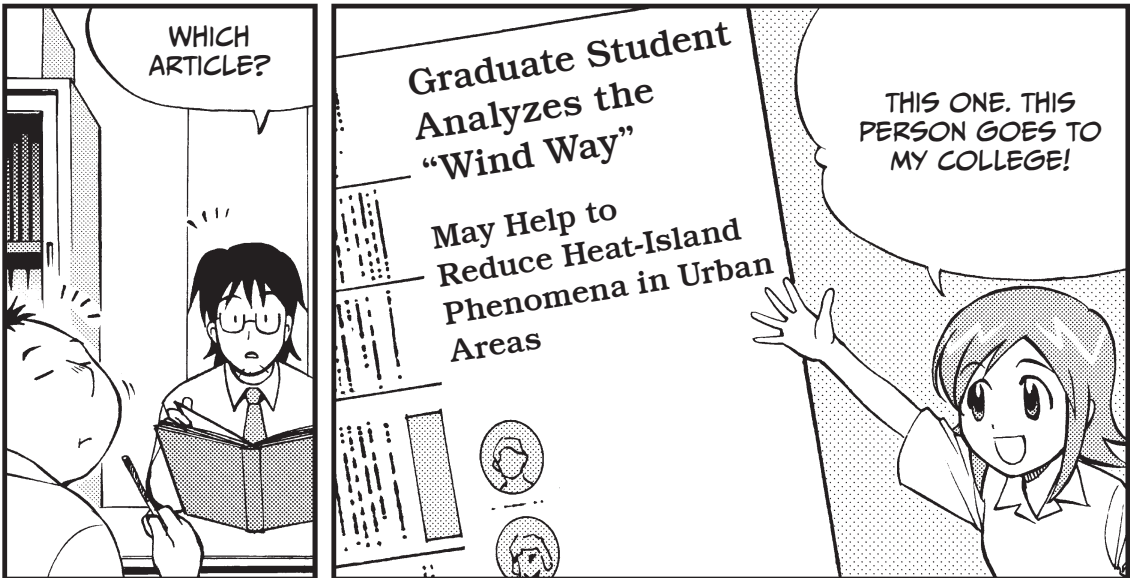
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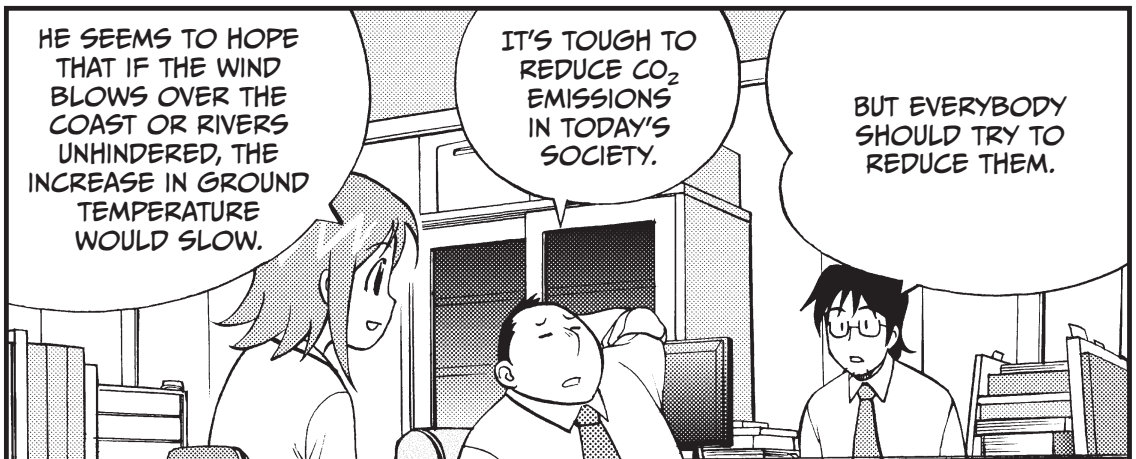
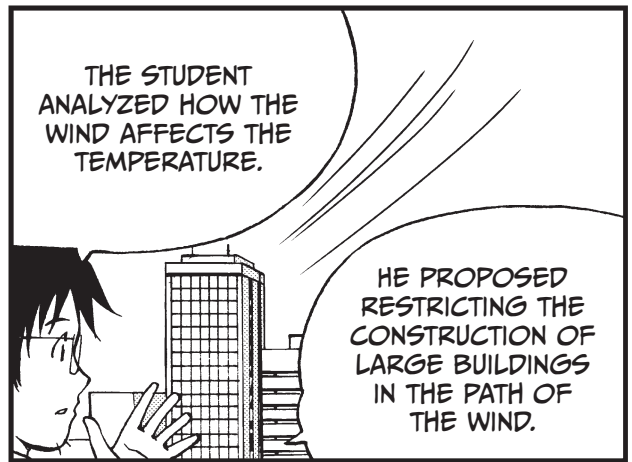
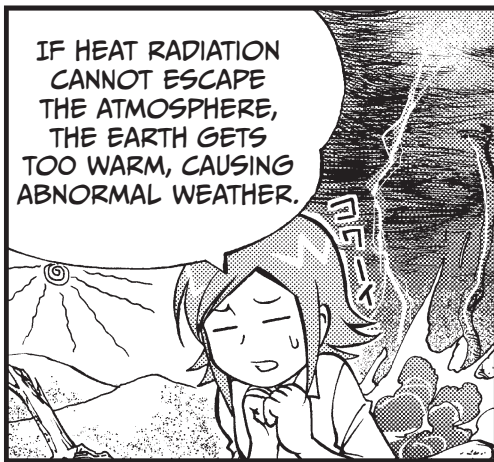
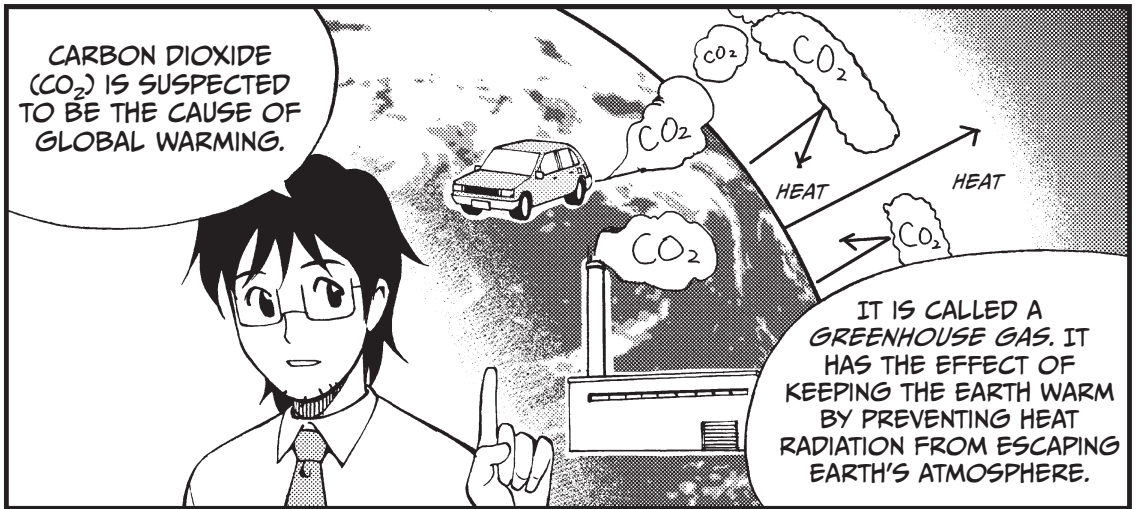
LET'S INTEGRATE A FUNCTION!

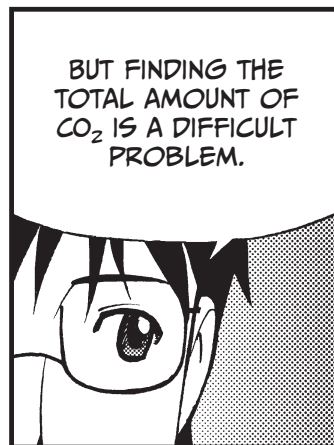
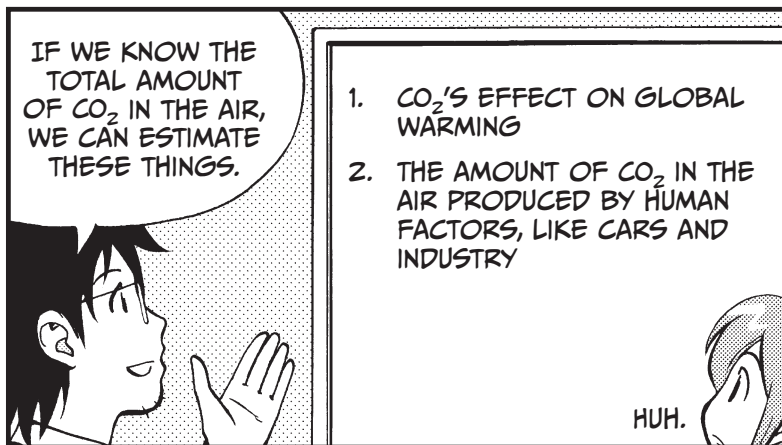
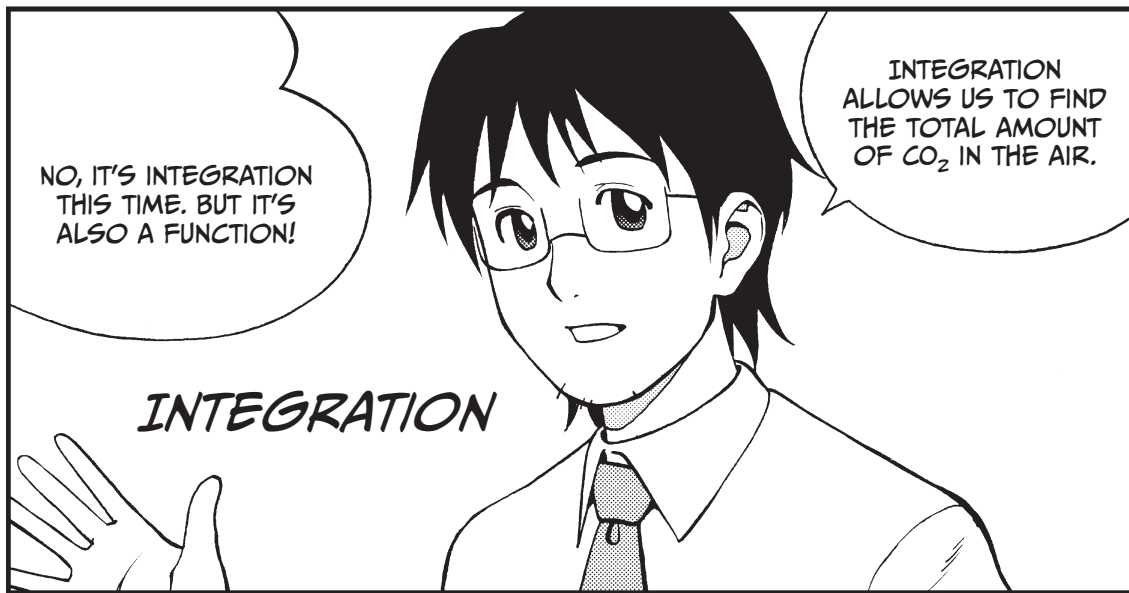
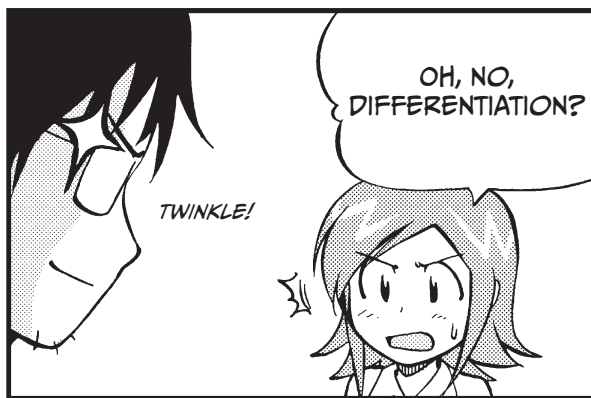


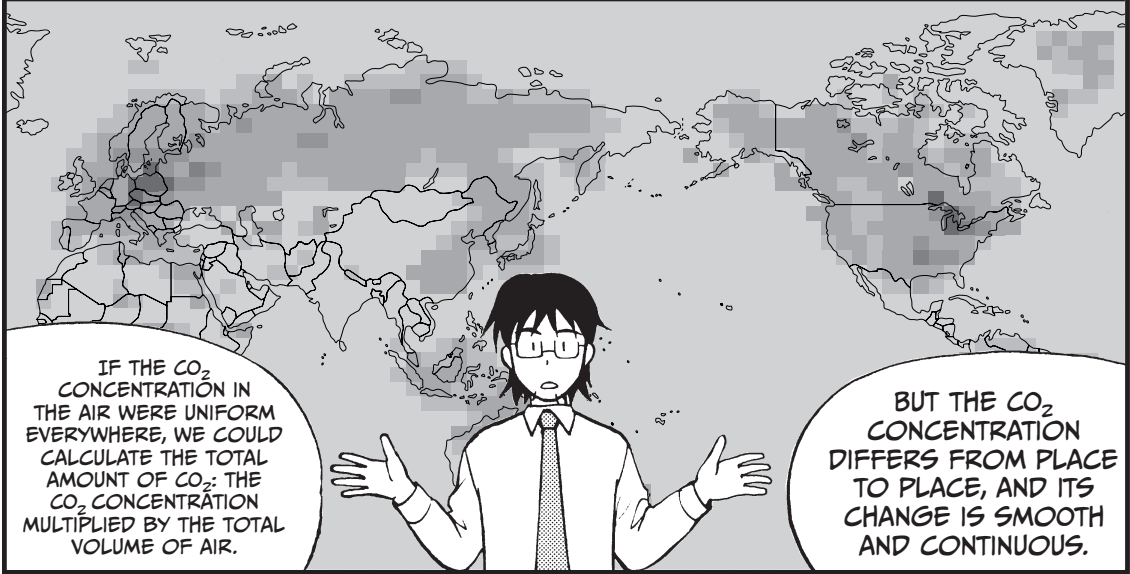


\* THE ASAGAKE TIMES



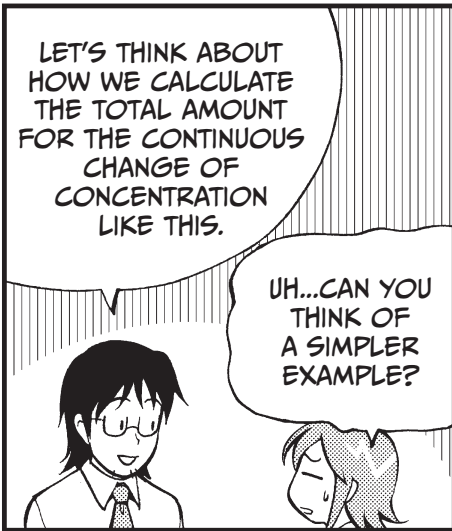






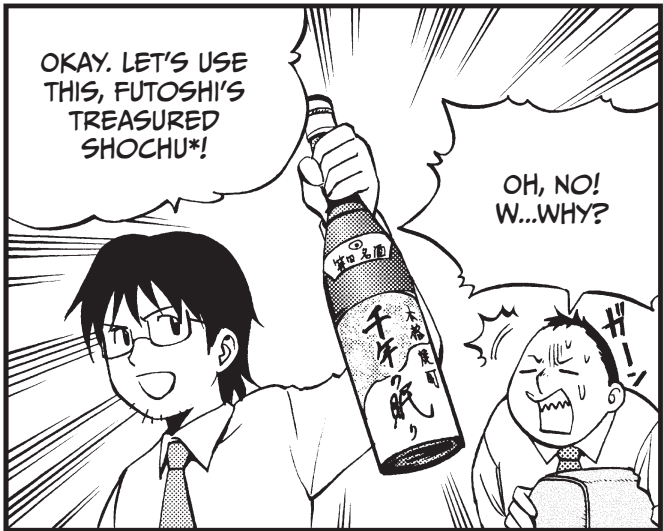
IF THE CO<sub>2</sub> CONCENTRATION IN THE AIR WERE UNIFORM EVERYWHERE, WE COULD CALCULATE THE TOTAL AMOUNT OF CO<sub>2</sub>: THE CO<sub>2</sub> CONCENTRATION MULTIPLIED BY THE TOTAL VOLUME OF AIR.

BUT THE CO<sub>2</sub> CONCENTRATION DIFFERS FROM PLACE TO PLACE, AND ITS CHANGE IS SMOOTH AND CONTINUOUS.



LET'S THINK ABOUT HOW WE CALCULATE THE TOTAL AMOUNT FOR THE CONTINUOUS CHANGE OF CONCENTRATION LIKE THIS.

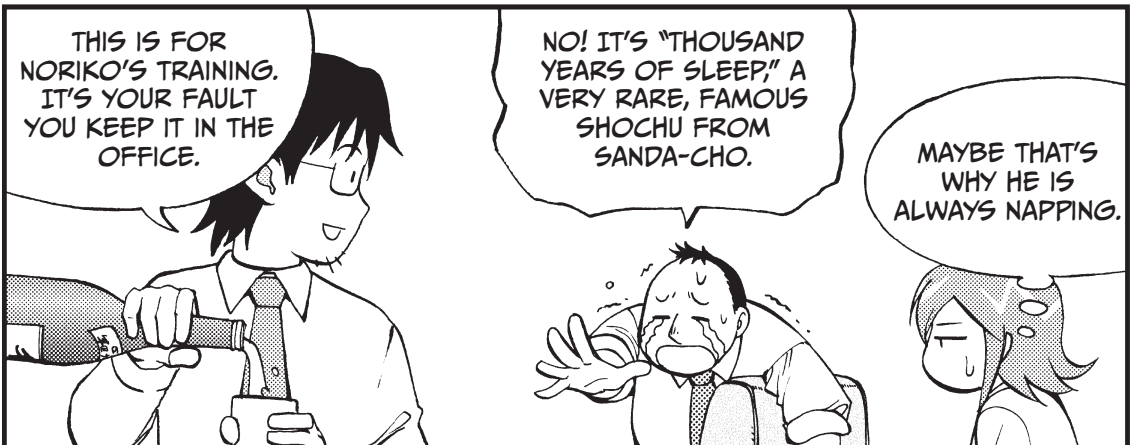
UH...CAN YOU THINK OF A SIMPLER EXAMPLE?



OKAY. LET'S USE THIS, FUTOSHI'S TREASURED SHOCHU\*!

OH, NO! W...WHY?

\* A JAPANESE DISTILLED SPIRIT

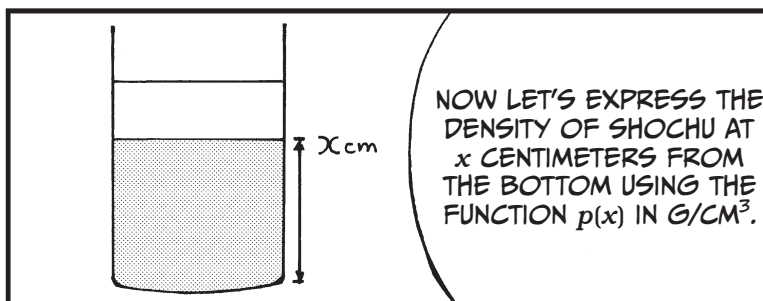
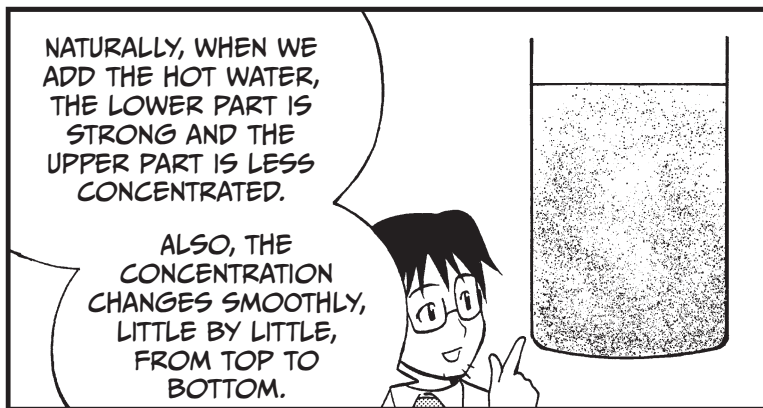
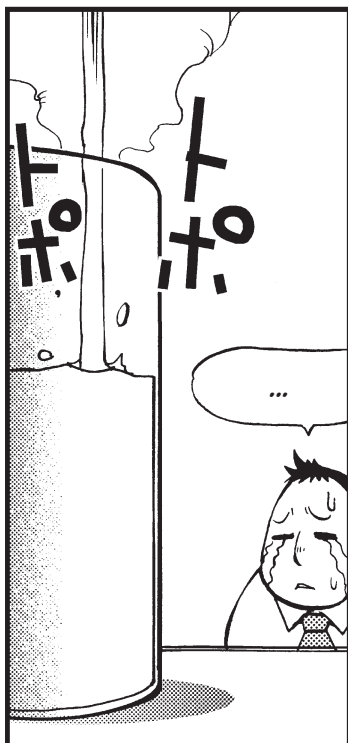
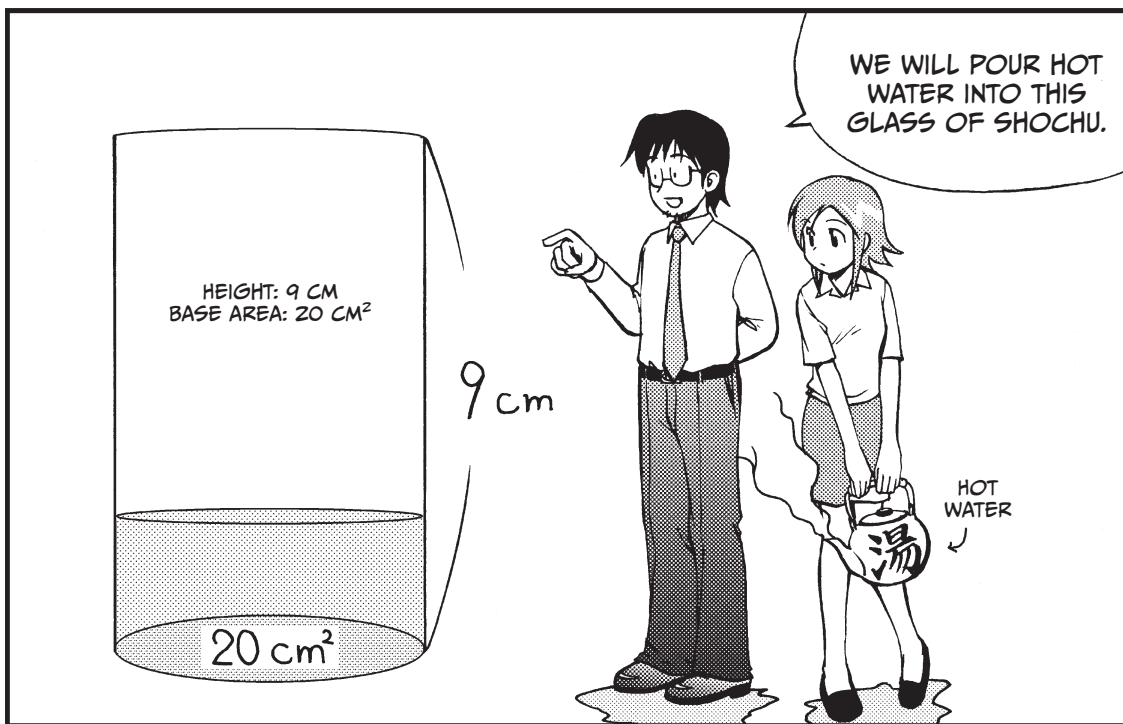


THIS IS FOR NORIKO'S TRAINING. IT'S YOUR FAULT YOU KEEP IT IN THE OFFICE.

NO! IT'S "THOUSAND YEARS OF SLEEP," A VERY RARE, FAMOUS SHOCHU FROM SANDA-CHO.

MAYBE THAT'S WHY HE IS ALWAYS NAPPING.

# ILLUSTRATING THE FUNDAMENTAL THEOREM OF CALCULUS



Density  $p(x)$

2

0.02

0 Height 9  $x$

SUPPOSE  $p(x)$  IS EXPRESSED AS

$$p(x) = \frac{2}{(x+1)^2}$$

NOW NORIKO, WHAT IS THE AMOUNT OF ALCOHOL IN GRAMS CONTAINED IN THIS SHOCHU WITH HOT WATER?

I CAN'T FIGURE IT OUT THAT QUICKLY.

**STEP 1—WHEN THE DENSITY IS CONSTANT**

BUT IF THE DENSITY IS CONSTANT, IT'S EASY. THE TOTAL AMOUNT OF ALCOHOL EQUALS THE DENSITY MULTIPLIED BY THE VOLUME OF THE CONTAINER.

$p(x)$

0.1

9  $x$

IF THE DENSITY IS  $0.1 \text{ G/CM}^3$ , AS SHOWN IN THIS GRAPH, WE NEED TO CALCULATE THE DENSITY TIMES THE HEIGHT TIMES THE BASE AREA:  $0.1 \times 9 \times 20 = 18 \text{ GRAMS}$ , WHICH IS THE AMOUNT OF ALCOHOL.

ISN'T IT THE SAME AS CALCULATING THE AREA OF THE SHADED PART OF THE GRAPH?

$p(x)$

0.1

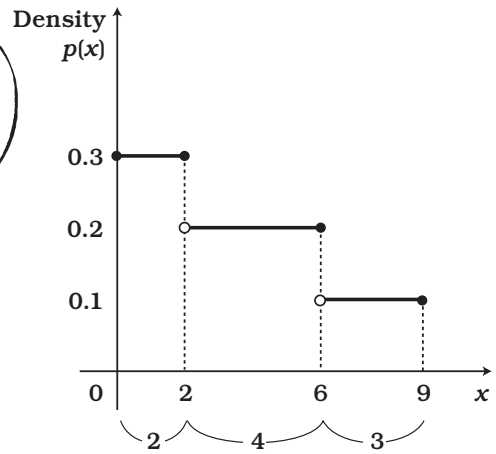
9  $x$

YOU ARE RIGHT! BUT TO GET THE VOLUME, WE MUST ALSO MULTIPLY HEIGHT  $x$  BY THE BASE AREA,  $20 \text{ CM}^2$ .

## STEP 2—WHEN THE DENSITY CHANGES STEPWISE

NOW, LET'S IMAGINE A GLASS OF SHOCHU WHERE THE DENSITY CHANGES STEPWISE,

AS REPRESENTED BY THIS GRAPH, FOR EXAMPLE.



WHY DON'T YOU CALCULATE IT, NORIKO?

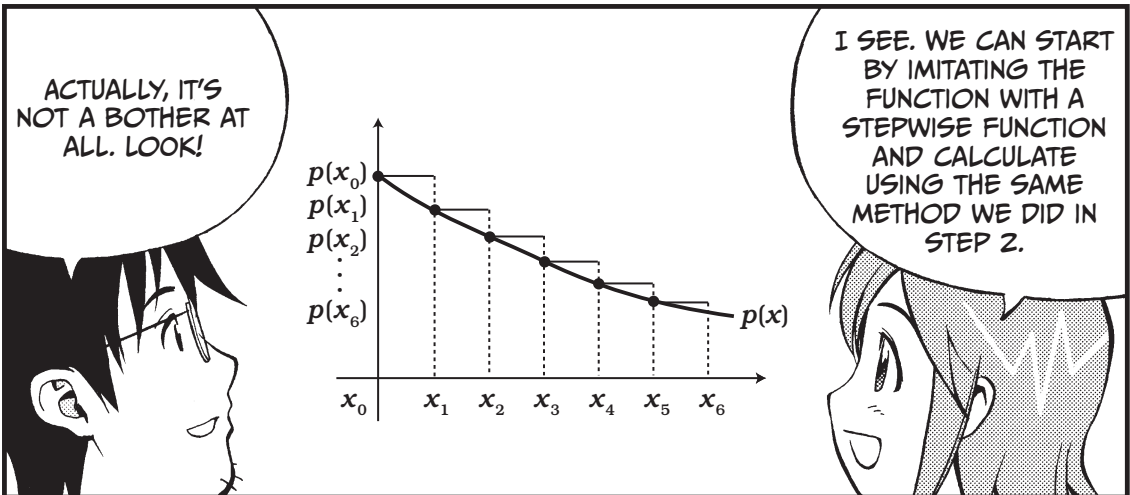
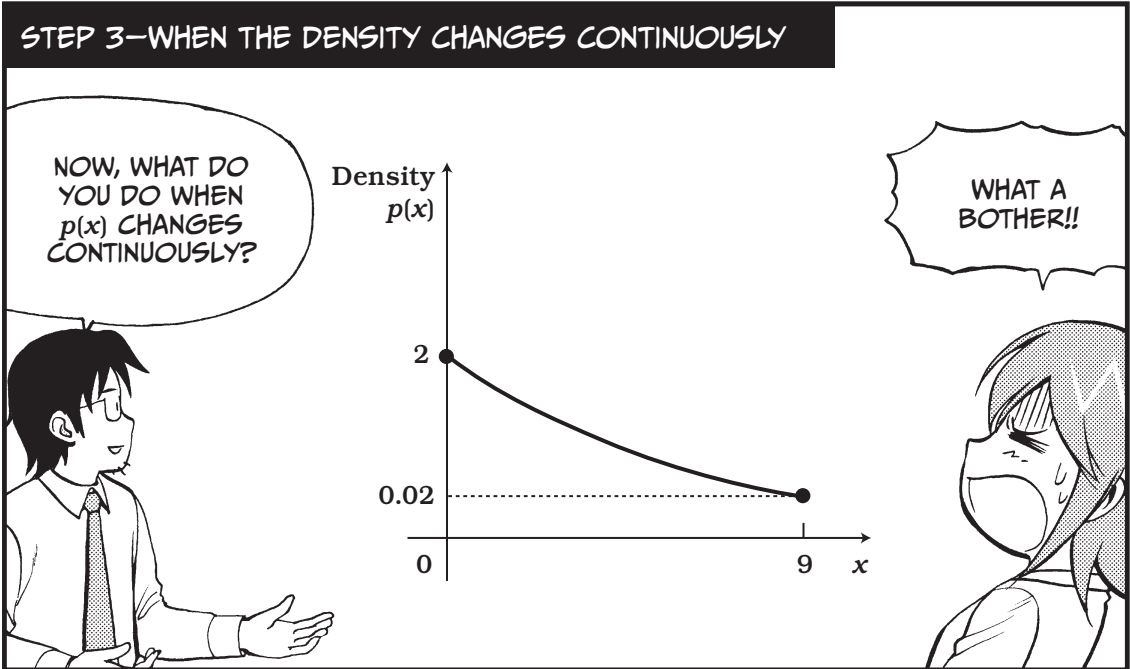
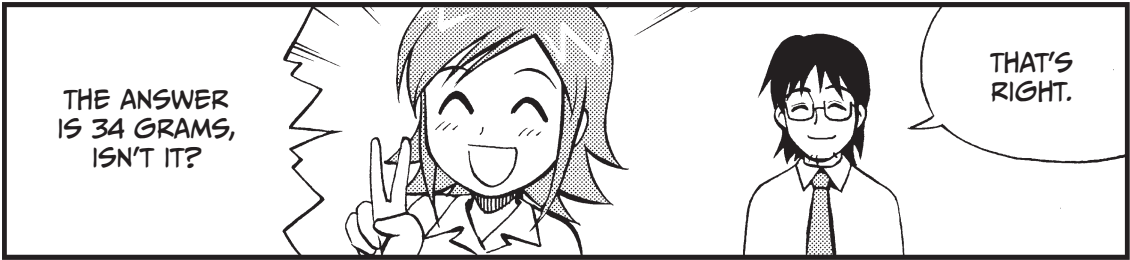
WELL, SEPARATING THE GRAPH INTO THE STEPS...THE BASE AREA IS 20 CM<sup>2</sup>...

$$0.3 \times 2 \times 20 + 0.2 \times 4 \times 20 + 0.1 \times 3 \times 20$$

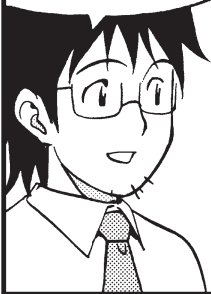
$$\left( \begin{array}{l} \text{Alcohol for} \\ \text{the portion of} \\ 0 \leq x \leq 2 \end{array} \right) \left( \begin{array}{l} \text{Alcohol for} \\ \text{the portion of} \\ 2 < x \leq 6 \end{array} \right) \left( \begin{array}{l} \text{Alcohol for} \\ \text{the portion of} \\ 6 < x \leq 9 \end{array} \right)$$

$$= (0.3 \times 2 + 0.2 \times 4 + 0.1 \times 3) \times 20 = 34$$

SO...



RIGHT! DIVIDING  
THE X-AXIS AT  $x_0$ ,  
 $x_1$ ,  $x_2$ , ..., AND  $x_6$ ,



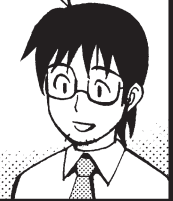
The density is constant between  $x_0$  and  $x_1$  and is  $p(x_0)$ .

The density is constant between  $x_1$  and  $x_2$  and is  $p(x_1)$ .

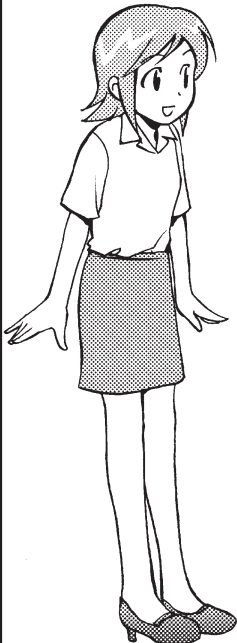
The density is constant between  $x_2$  and  $x_3$  and is  $p(x_2)$ .

IN THIS WAY, WE IMITATE  
 $p(x)$  WITH A STEPWISE  
FUNCTION.

CALCULATING  
THE AMOUNT OF  
ALCOHOL WITH THIS  
STEPWISE FUNCTION  
GIVES US AN AMOUNT  
IMITATING THE  
EXACT AMOUNT OF  
ALCOHOL.

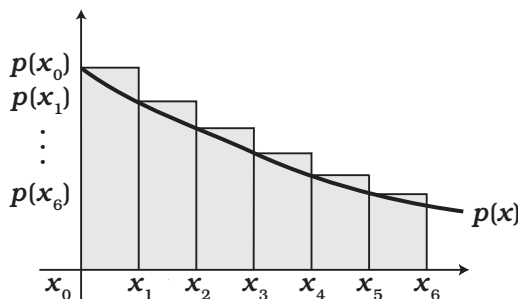


THAT'S THIS  
CALCULATION,  
ISN'T IT?



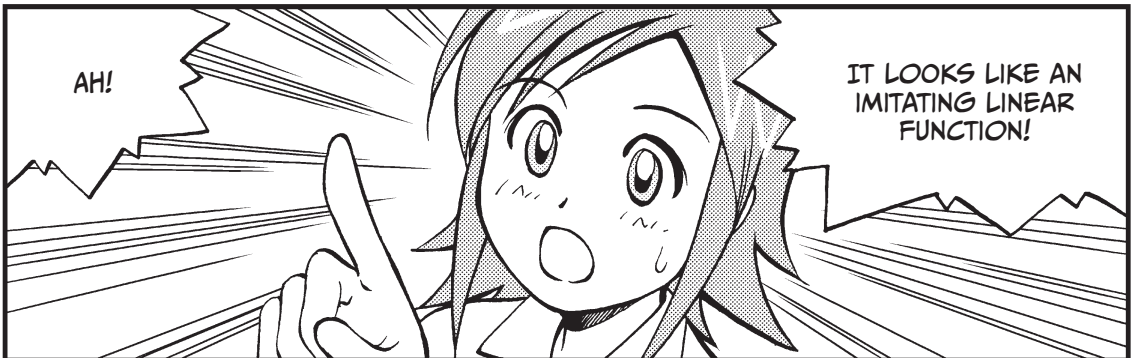
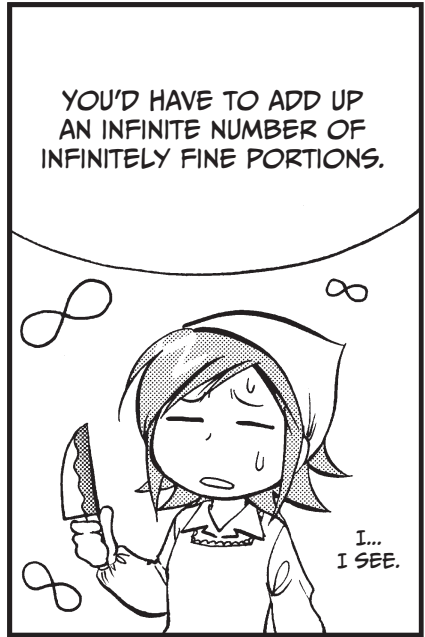
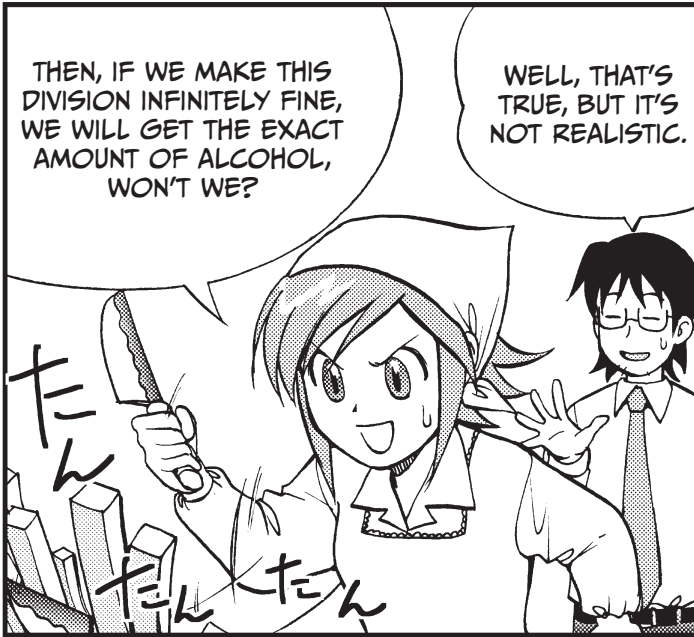
$$\begin{aligned}
 & p(x_0) \times (x_1 - x_0) \times 20 \\
 & p(x_1) \times (x_2 - x_1) \times 20 \\
 & p(x_2) \times (x_3 - x_2) \times 20 \\
 & p(x_3) \times (x_4 - x_3) \times 20 \\
 & p(x_4) \times (x_5 - x_4) \times 20 \\
 & + p(x_5) \times (x_6 - x_5) \times 20
 \end{aligned}$$

Approximate  
amount of alcohol



RIGHT. THE SHADED  
AREA OF THE  
STEPWISE FUNCTION  
IS THE SUM OF THESE  
EXPRESSIONS (BUT  
WITHOUT MULTIPLYING  
BY 20 CM<sup>2</sup>, THE  
BASE AREA).





## STEP 4—REVIEW OF THE IMITATING LINEAR FUNCTION

When the derivative of  $f(x)$  is given by  $f'(x)$ , we had  $f(x) \approx f'(a)(x - a) + f(a)$  near  $x = a$ .

Transposing  $f(a)$ , we get

$$\textcircled{1} \quad f(x) - f(a) \approx f'(a)(x - a)$$

$$\text{or (Difference in } f) \approx (\text{Derivative of } f) \times (\text{Difference in } x)$$

If we assume that the interval between two consecutive values of  $x_0, x_1, x_2, x_3, \dots, x_6$  is small enough,  $x_1$  is close to  $x_0$ ,  $x_2$  is close to  $x_1$ , and so on.

Now, let's introduce a new function,  $q(x)$ , whose derivative is  $p(x)$ . This means  $q'(x) = p(x)$ .

Using  $\textcircled{1}$  for this  $q(x)$ , we get

$$(\text{Difference in } q) \approx (\text{Derivative of } q) \times (\text{Difference in } x)$$

$$q(x_1) - q(x_0) \approx p(x_0)(x_1 - x_0)$$

$$q(x_2) - q(x_1) \approx p(x_1)(x_2 - x_1)$$

The sum of the right sides of these expressions is the same as the sum of the left sides.

Some terms in the expressions for the sum cancel each other out.

~~$$q(x_1) - q(x_0) \approx p(x_0)(x_1 - x_0)$$~~

~~$$q(x_2) - q(x_1) \approx p(x_1)(x_2 - x_1)$$~~

~~$$q(x_3) - q(x_2) \approx p(x_2)(x_3 - x_2)$$~~

~~$$q(x_4) - q(x_3) \approx p(x_3)(x_4 - x_3)$$~~

~~$$q(x_5) - q(x_4) \approx p(x_4)(x_5 - x_4)$$~~

~~$$+ q(x_6) - q(x_5) \approx p(x_5)(x_6 - x_5)$$~~

$$q(x_6) - q(x_0) \approx \text{The sum}$$

SO WE NEED TO FIND  
FUNCTION  $q(x)$  THAT  
SATISFIES  $q'(x) = p(x)$ .

Substituting  $x_6 = 9$  and  $x_0 = 0$ , we get

The approximate amount of alcohol = the sum  $\times 20$

$$\{q(x_6) - q(x_0)\} \times 20$$

$$\{q(9) - q(0)\} \times 20$$



## STEP 5—APPROXIMATION → EXACT VALUE

WE HAVE JUST OBTAINED THE FOLLOWING RELATIONSHIP OF EXPRESSIONS SHOWN IN THE DIAGRAM.



The approximate amount of alcohol (÷ 20) given by the stepwise function:

$$p(x_0)(x_1 - x_0) + p(x_1)(x_2 - x_1) + \dots$$

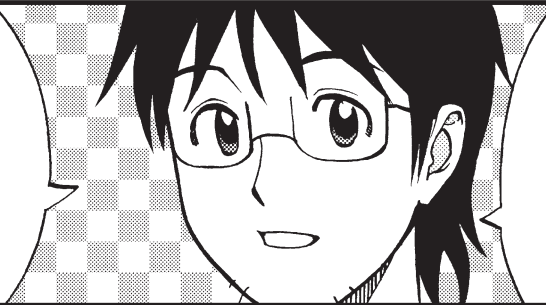
$$\stackrel{\textcircled{2}}{\approx} q(9) - q(0)$$

(Constant)

① ≈

The exact amount of alcohol (÷ 20)

BUT IF WE INCREASE THE NUMBER OF POINTS  $x_0, x_1, x_2, x_3,$  AND SO ON, UNTIL IT BECOMES INFINITE,



WE CAN SAY THAT RELATIONSHIP ① CHANGES FROM "APPROXIMATION" TO "EQUALITY."

BUT, SINCE THE SUM OF THE EXPRESSIONS HAVE BEEN IMITATING THE CONSTANT VALUE  $q(9) - q(0),$



The sum of  $p(x_i)(x_{i+1} - x_i)$  for an infinite number of  $x_i$

$$= q(9) - q(0)$$

≈

The exact amount of alcohol (÷ 20)

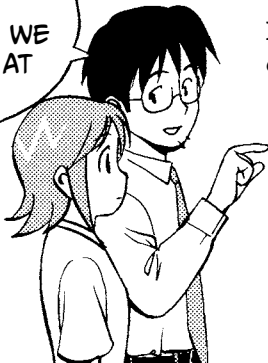
≈

WE GET THE RELATIONSHIP SHOWN HERE.\*

\* WE WILL OBTAIN THIS RELATIONSHIP MORE RIGOROUSLY ON PAGE 94.

**STEP 6— $p(x)$  IS THE DERIVATIVE OF  $q(x)$**

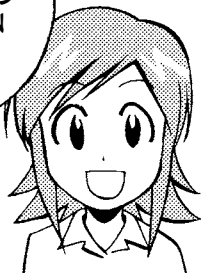
NOW NORIKO,  
THE NEXT  
EXPRESSION WE  
WILL LOOK AT  
IS THIS.



$$\text{If we suppose } q(x) = -\frac{2}{x+1}, \text{ then } q'(x) = \frac{2}{(x+1)^2} = p(x)$$

In other words,  $p(x)$  is the derivative of  $q(x)$ .  
 $q(x)$  is called the *antiderivative* of  $p(x)$ .

SO, THIS  $q(x)$  IS  
THE FUNCTION  
WE WANTED.



The amount of alcohol

$$\begin{aligned} &= \{q(9) - q(0)\} \times 20 \\ &= \left\{ -\frac{2}{9+1} - \left( -\frac{2}{0+1} \right) \right\} \times 20 \\ &= 36 \text{ grams} \end{aligned}$$

THE AMOUNT OF  
ALCOHOL IN A GLASS  
OF SHOCHU WITH HOT  
WATER IS GENERALLY  
24.3 GRAMS.

SO, WE  
HAVE A VERY  
STRONG  
DRINK HERE.



36  
⋮



SINCE THE SUM  
OF INFINITE  
TERMS WE HAVE  
BEEN DOING



REQUIRES A LOT  
OF TIME TO WRITE  
DOWN, I WILL  
SHOW YOU ITS  
SYMBOL.

# USING THE FUNDAMENTAL THEOREM OF CALCULUS

$$p(x_0)(x_1 - x_0) + p(x_1)(x_2 - x_1) + \dots + p(x_5)(x_6 - x_5)$$

THE ABOVE  
EXPRESSION

$$\sum p(x) \Delta x$$

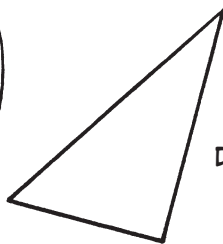
$$x = x_0, x_1, \dots, x_5$$

CAN BE WRITTEN  
IN THIS WAY.

OH, SIMPLE!

BUT, WHAT IS  $\Delta$ ?

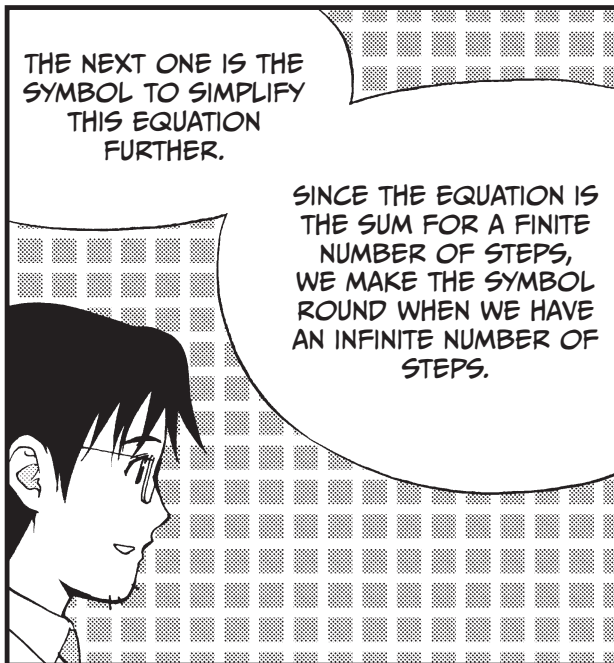
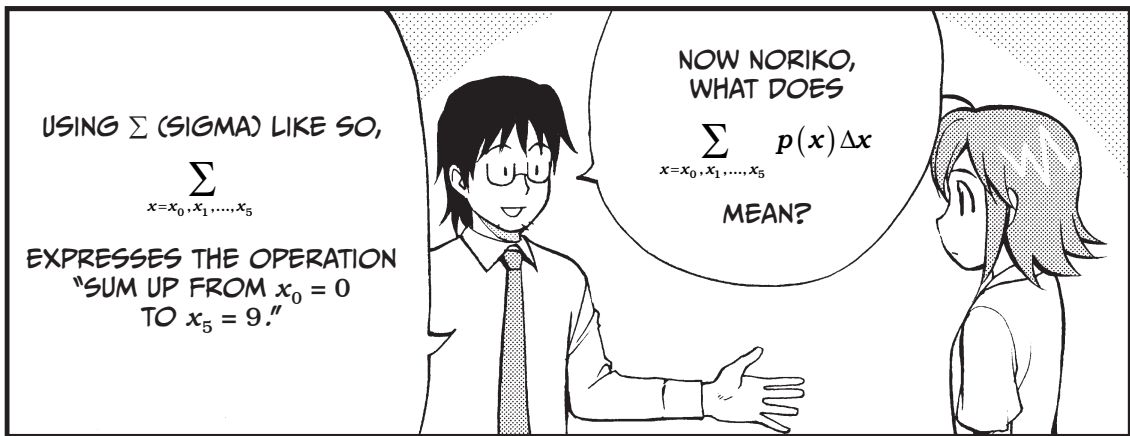
$\Delta$  (DELTA) IS A GREEK  
LETTER. THE SYMBOL IS  
USED TO EXPRESS THE  
AMOUNT OF CHANGE.



DELTA

THIS  $\Delta x$  EXPRESSES THE  
DISTANCE TO THE NEXT  
POINT. IN OTHER WORDS,  
IT IS, FOR EXAMPLE,  
 $(x_1 - x_0)$  OR  $(x_2 - x_1)$ .

WHAT ABOUT  $\Sigma$ ?



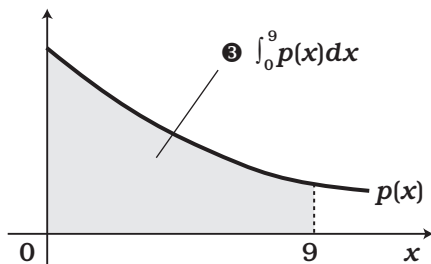
$$\sum p(x)\Delta x \rightarrow \int_0^9 p(x)\Delta x \rightarrow \int_0^9 p(x)dx$$

I EXPAND  $\Sigma$  TO MAKE  $\int$ , AND

YANK!

REPLACE  $\Delta$  WITH  $d$ .

BOY!



EXPRESSION ③ MEANS THE SUM WHEN THE INTERVAL IS MADE INFINITELY SMALL, AND IT EXPRESSES THE AREA BETWEEN THE GRAPH ON THE LEFT AND THE X-AXIS.

THIS IS CALLED A DEFINITE INTEGRAL.

IF WE KNOW  $p(x)$  IS THE DERIVATIVE OF  $q(x)$ ,

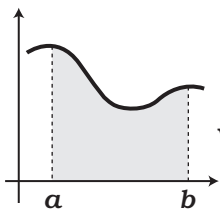
$$\int_a^b p(x) dx = q(b) - q(a)$$

WE HAVE CALCULATED THE SUM EXTREMELY EASILY IN THIS WAY, HAVEN'T WE?

DEFINITE INTEGRAL, YOU ARE WONDERFUL!

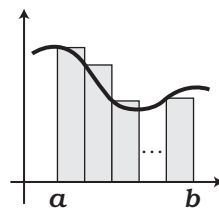
NOT NEARLY AS EXCITED

### SUMMARY



$$p(x) = \int_a^b p(x) dx \approx \sum_{x=x_0, x_1, \dots, x_5} p(x)\Delta x = q(b) - q(a)$$

We must find  $q(x)$  that satisfies  $q'(x) = p(x)$ .



THIS IS THE FUNDAMENTAL THEOREM OF CALCULUS!

## A STRICT EXPLANATION OF STEP 5

In the explanation given before (page 89), we used, as the basic expression,  $q(x_1) - q(x_0) \approx p(x_0)(x_1 - x_0)$ , a “crude” expression which roughly imitates the exact expression. For those who think this is a sloppy explanation, we will explain more carefully here. Using the mean value theorem, we can reproduce the same result.

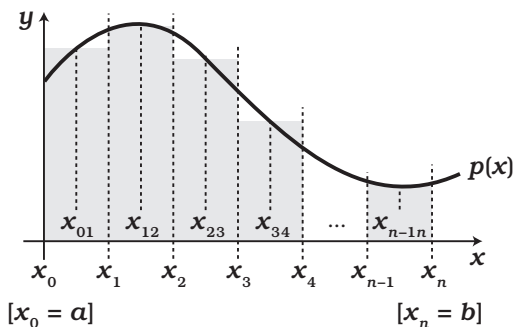


We first find  $q(x)$  that satisfies  $q'(x) = p(x)$ .

We place points  $x_0 (= a)$ ,  $x_1$ ,  $x_2$ ,  $x_3$ , ...,  $x_n (= b)$  on the  $x$ -axis.

We then find point  $x_{01}$  that exists between  $x_0$  and  $x_1$  and satisfies  $q(x_1) - q(x_0) \approx q'(x_{01})(x_1 - x_0)$ .

The existence of such a point is guaranteed by the mean value theorem. Similarly, we find  $x_{12}$  between  $x_1$  and  $x_2$  and get



$$q(x_2) - q(x_1) \approx q'(x_{12})(x_2 - x_1)$$

Repeating this operation, we get

$$\begin{array}{l}
 q(x_1) - q(x_0) = q'(x_{01})(x_1 - x_0) = p(x_{01})(x_1 - x_0) \\
 q(x_2) - q(x_1) = q'(x_{12})(x_2 - x_1) = p(x_{12})(x_2 - x_1) \\
 q(x_3) - q(x_2) = q'(x_{23})(x_3 - x_2) = p(x_{23})(x_3 - x_2) \\
 \dots \\
 + q(x_n) - q(x_{n-1}) = q'(x_{n-1n})(x_n - x_{n-1}) = p(x_{n-1n})(x_n - x_{n-1}) \\
 \hline
 q(x_n) - q(x_0) \leftarrow \text{Always equal} \rightarrow \text{Approximate area} \\
 \downarrow \qquad \qquad \qquad \downarrow \text{Infinitely fine sections} \\
 q(b) - q(a) \leftarrow \text{Equal} \rightarrow \text{Exact area}
 \end{array}$$

Summing up

Areas of these steps

This corresponds to the diagram in step 5.

## USING INTEGRAL FORMULAS

### FORMULA 3-1: THE INTEGRAL FORMULAS

$$\textcircled{1} \quad \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

The intervals of definite integrals of the same function can be joined.

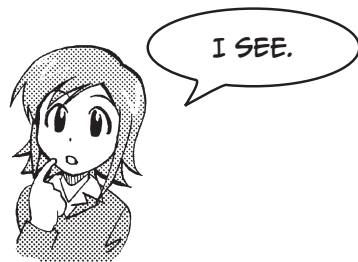
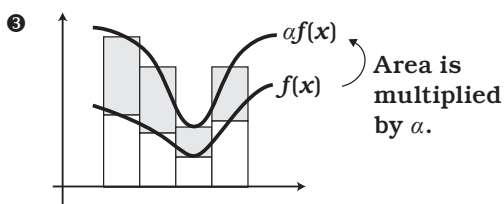
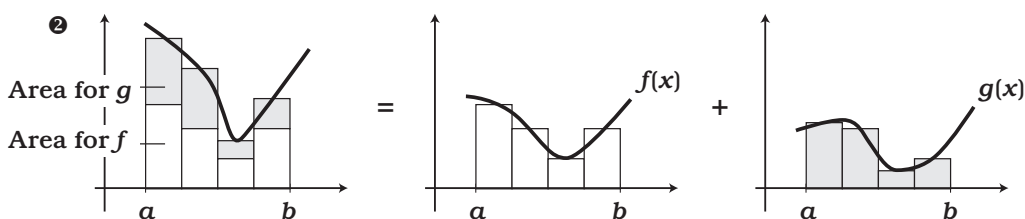
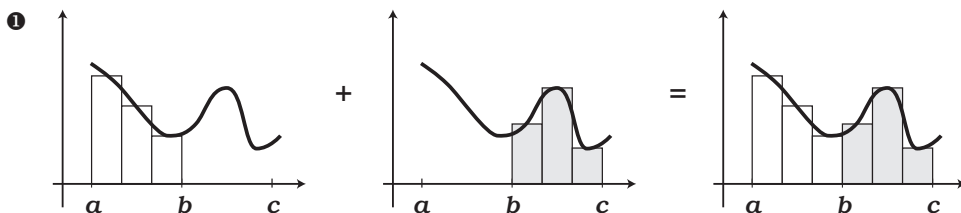
$$\textcircled{2} \quad \int_a^b \{f(x) + g(x)\} dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

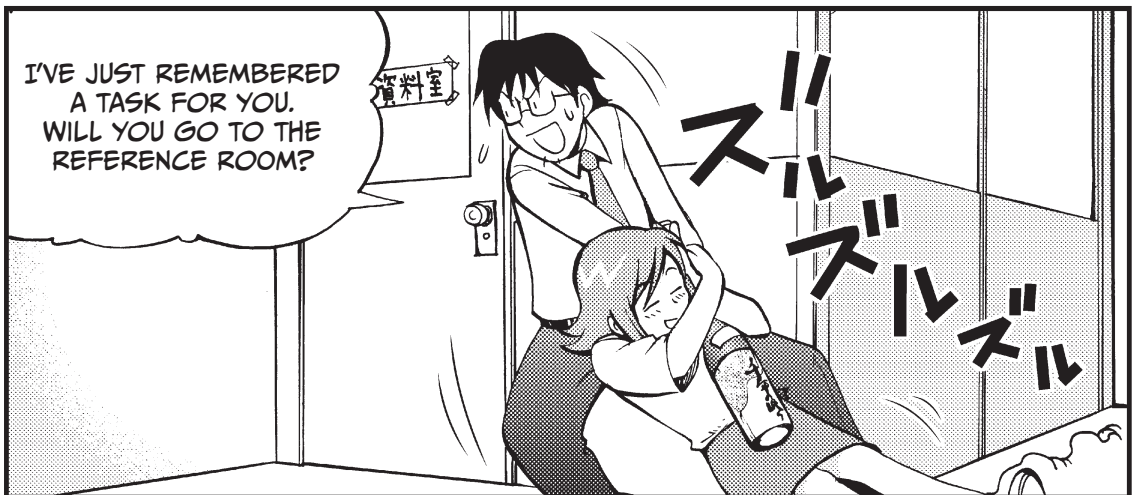
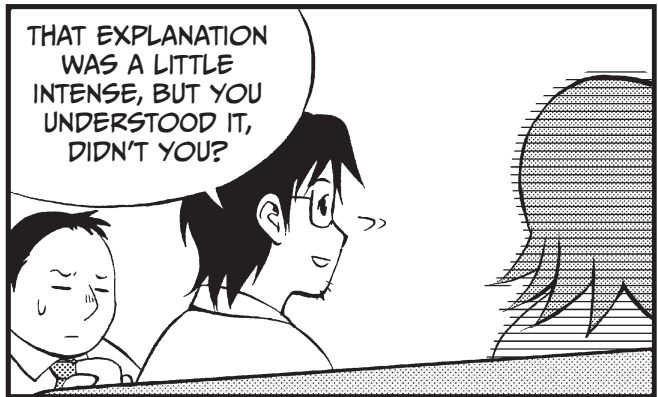
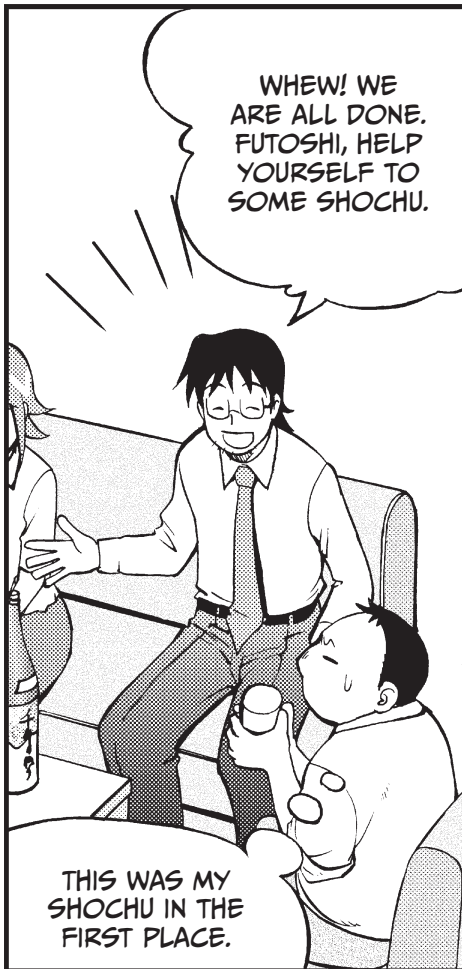
A definite integral of a sum can be divided into the sum of definite integrals.

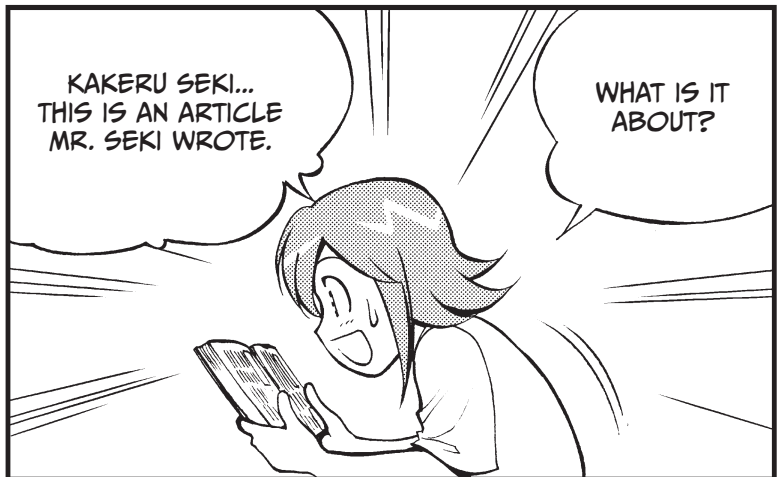
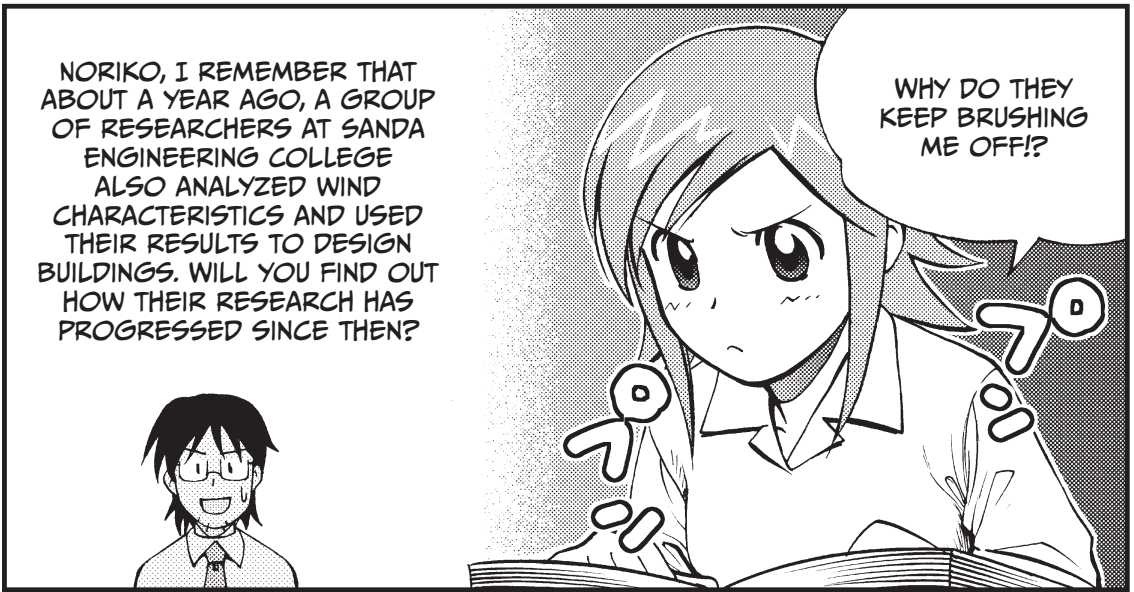
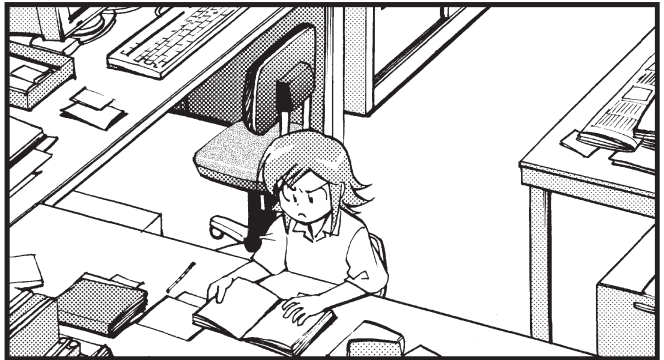
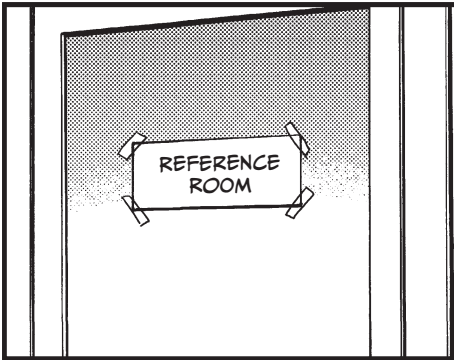
$$\textcircled{3} \quad \int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx$$

The multiplicative constant within a definite integral can be moved outside the integral.

Expressions  $\textcircled{1}$  through  $\textcircled{3}$  can be understood intuitively if we draw their figures.



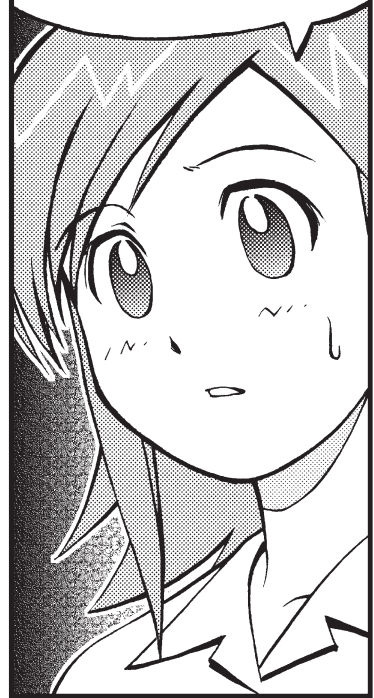




# Pollution in the Bay

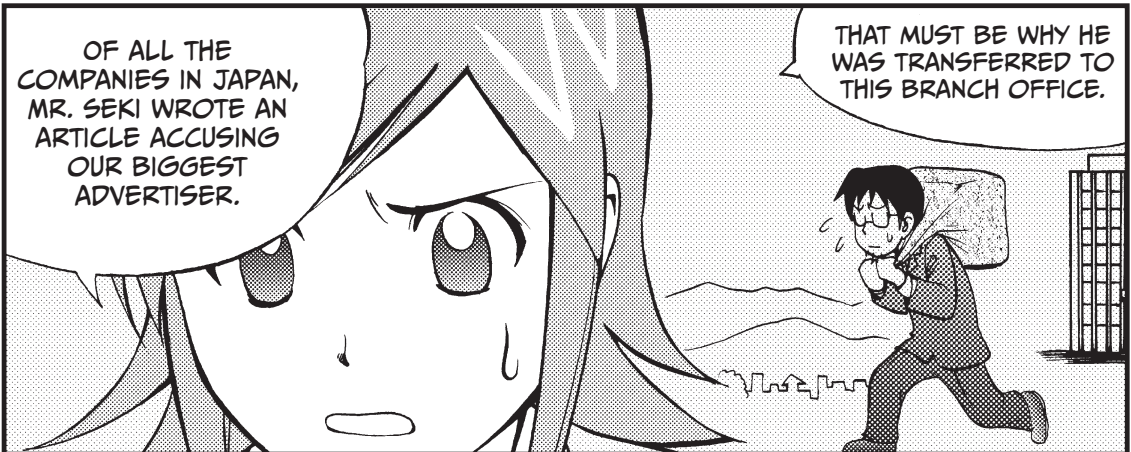
Waste Runoff from  
Burnham Chemical  
Products Is the  
Cause

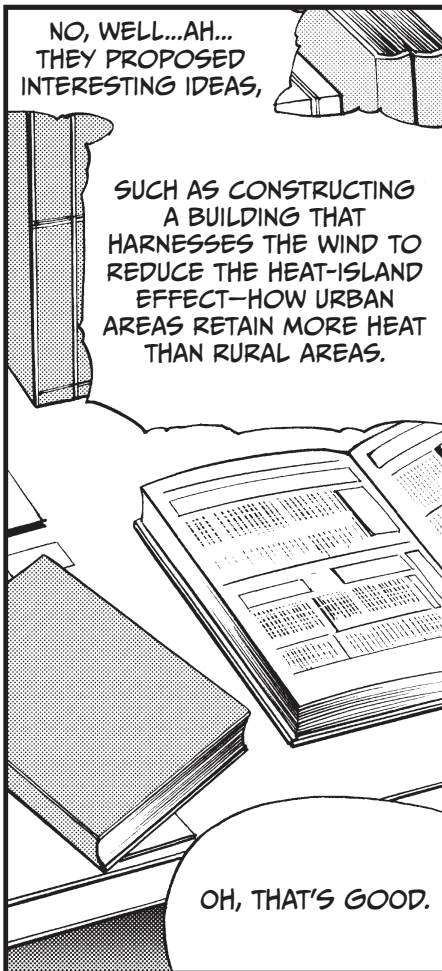
BURNHAM...  
THEY'RE ONE OF THE  
SPONSORS OF THE  
ASAGAKE TIMES.

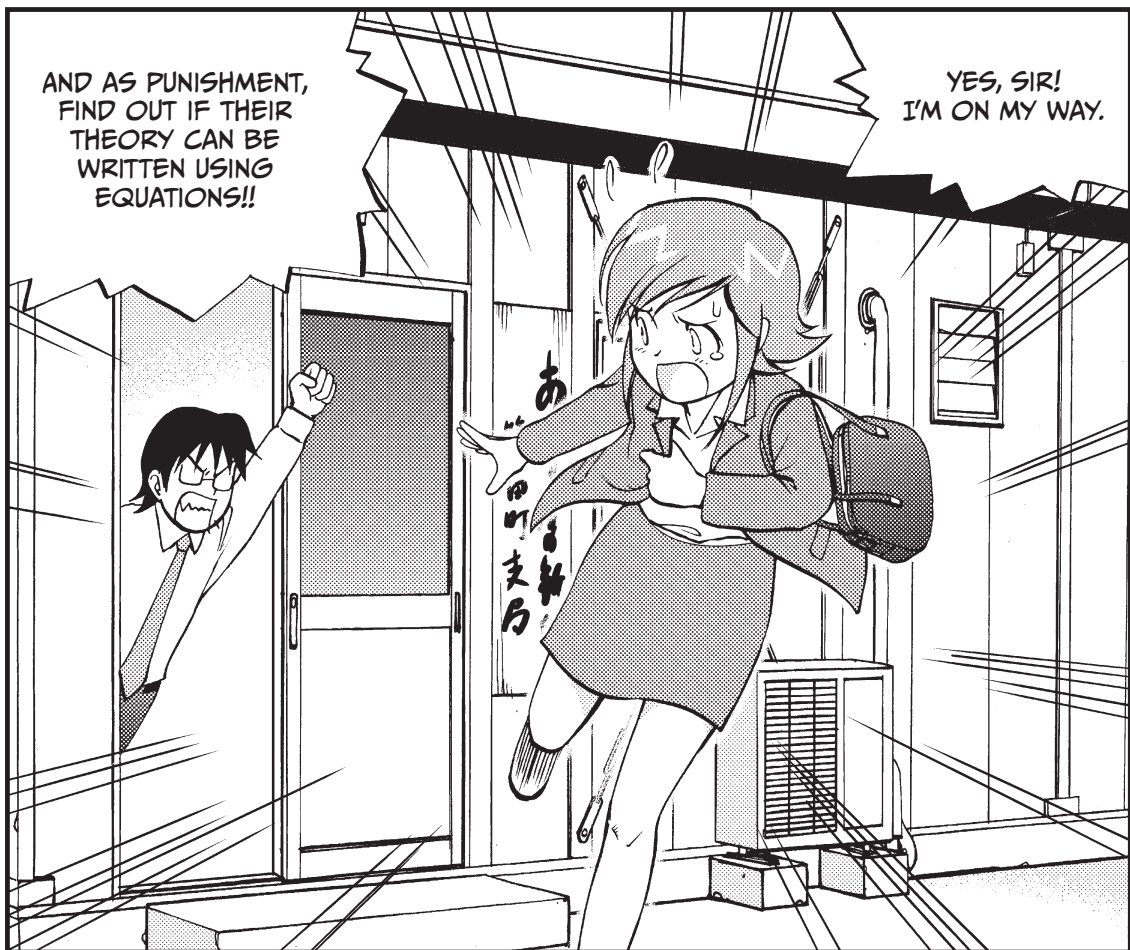
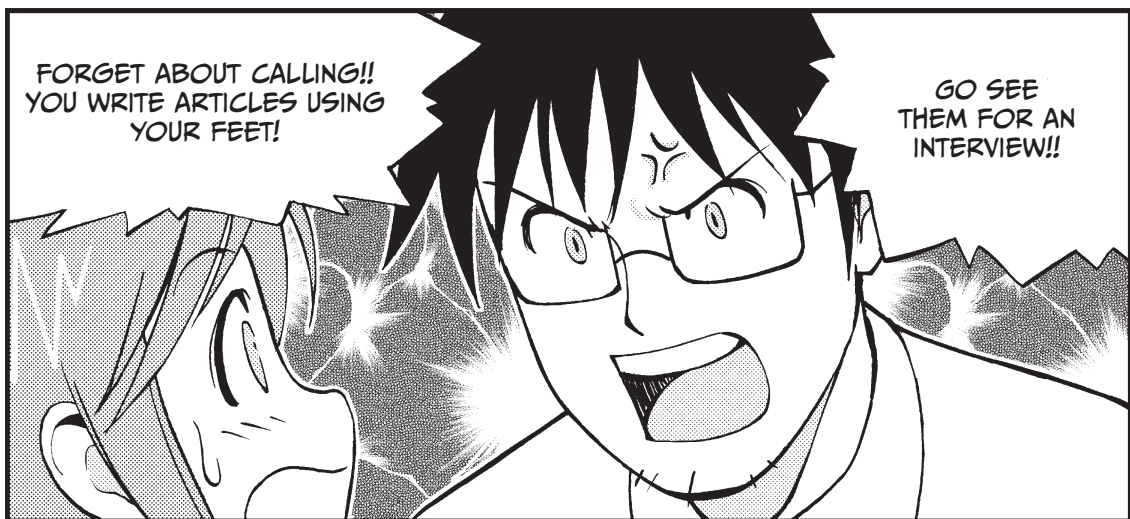


OF ALL THE  
COMPANIES IN JAPAN,  
MR. SEKI WROTE AN  
ARTICLE ACCUSING  
OUR BIGGEST  
ADVERTISER.

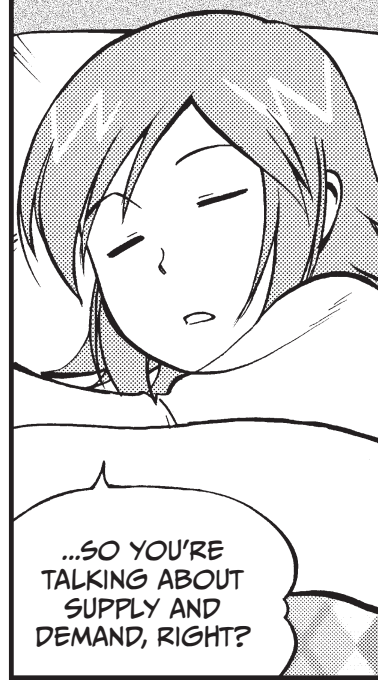
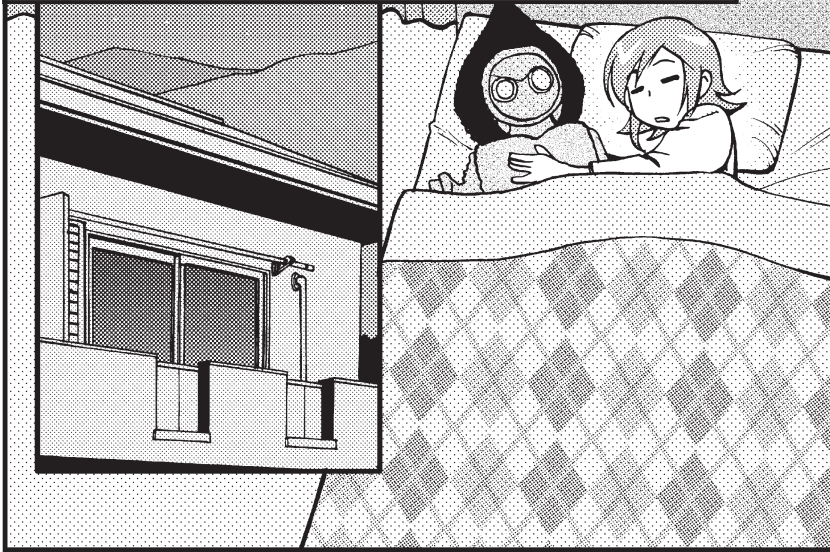
THAT MUST BE WHY HE  
WAS TRANSFERRED TO  
THIS BRANCH OFFICE.



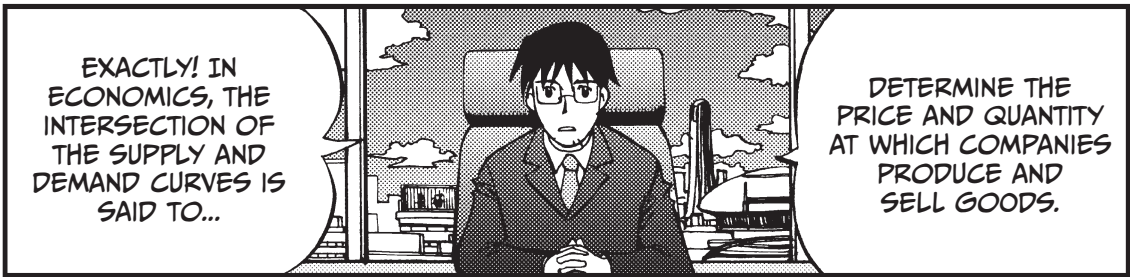




# APPLYING THE FUNDAMENTAL THEOREM

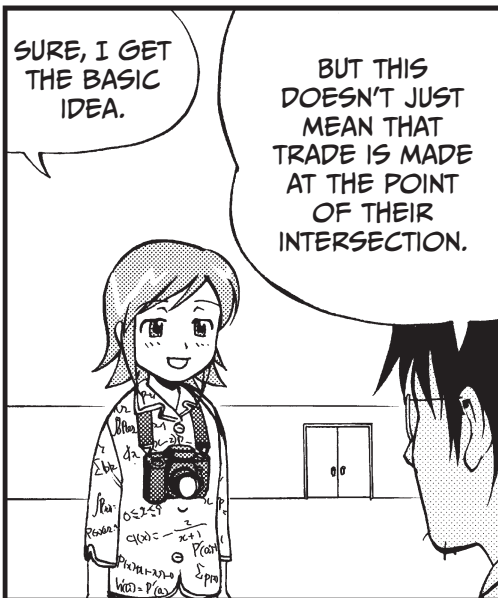


...SO YOU'RE TALKING ABOUT SUPPLY AND DEMAND, RIGHT?



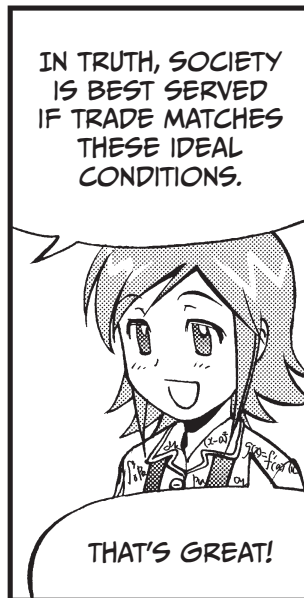
EXACTLY! IN ECONOMICS, THE INTERSECTION OF THE SUPPLY AND DEMAND CURVES IS SAID TO...

DETERMINE THE PRICE AND QUANTITY AT WHICH COMPANIES PRODUCE AND SELL GOODS.



SURE, I GET THE BASIC IDEA.

BUT THIS DOESN'T JUST MEAN THAT TRADE IS MADE AT THE POINT OF THEIR INTERSECTION.



IN TRUTH, SOCIETY IS BEST SERVED IF TRADE MATCHES THESE IDEAL CONDITIONS.

THAT'S GREAT!



YES, WE CAN EASILY UNDERSTAND WHY THIS IS TRUE USING THE FUNDAMENTAL THEOREM OF CALCULUS.

## SUPPLY CURVE



FIRST, LET'S CONSIDER HOW COMPANIES MAXIMIZE PROFIT IN A PERFECTLY COMPETITIVE MARKET. WE'LL TRY TO DERIVE A SUPPLY CURVE FIRST.

The profit  $P(x)$  when  $x$  units of a commodity are produced is given by the following function:

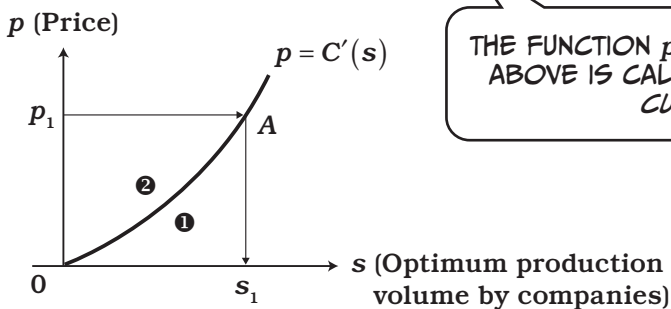
$$\begin{array}{ccccccc} \boxed{P(x)} & & \boxed{p} & & \boxed{x} & & \boxed{C(x)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\text{Profit}) = (\text{Price}) \times (\text{Production Quantity}) - (\text{Cost}) = px - C(x) \end{array}$$

where  $C(x)$  is the cost of production.

Let's assume the  $x$  value that maximizes the profit  $P(x)$  is the quantity of production  $s$ .

A company wants to maximize its profits. Recall that to find a function's extrema, we take the derivative and set it to zero. This means that the company's maximum profit occurs when

$$P'(s) = p - C'(s) = 0$$



THE FUNCTION  $p = C'(s)$  OBTAINED ABOVE IS CALLED THE SUPPLY CURVE!

Price  $p_1$  corresponds to point  $A$  on the function, which leads us to optimum production volume  $s_1$ .

The rectangle bounded by these four points ( $p_1$ ,  $A$ ,  $s_1$ , and the origin) equals the price multiplied by the production quantity. This should be the companies' gross profits, before subtracting their costs of production. But look, the area ❶ of this graph corresponds to the companies' production costs, and we can obtain it using an integral.

$$\int_0^{s_1} C'(s) ds = C(s_1) - C(0) = C(s_1) = \text{Costs}$$

We used  
the Fundamental  
Theorem here.

To simplify,  
we assume  
 $C(0) = 0$ .

This means we can easily find the companies' net profit, which is represented by area ❷ in the graph, or the area of the rectangle minus area ❶.

## DEMAND CURVE

Next, let's consider the maximum benefit for consumers.

When consumers purchase  $x$  units of a commodity, the benefit  $B(x)$  for them is given by the equation:

$$B(x) = \text{Total Value of Consumption} - (\text{Price} \times \text{Quantity}) = u(x) - px$$

where  $u(x)$  is a function describing the value of the commodity for all consumers.

Consumers will purchase the most of this commodity when  $B(x)$  is maximized.

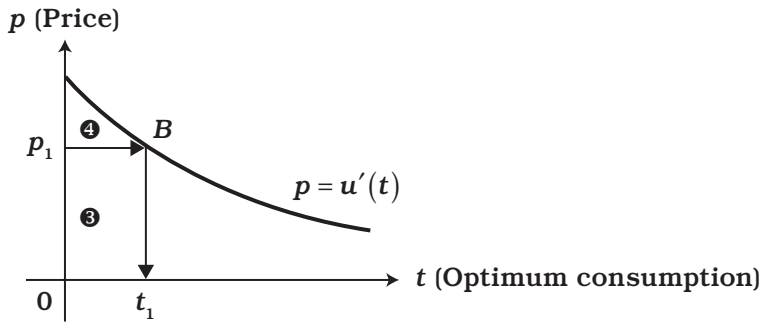
If we set the consumption value to  $t$  when the derivative of  $B(x) = 0$ , we get the following equation:\*

$$B'(t) = u'(t) - p = 0$$

THE FUNCTION  $p = u'(t)$  OBTAINED HERE IS CALLED THE DEMAND CURVE.



\* Again, you can see we're looking for extrema (where  $B'(t) = 0$ ), as consumers want to maximize their benefits.



So let's consider the area of the rectangle labeled ③, above, which corresponds to the price multiplied by the product consumption. In other words, this is the total amount consumers pay for a product.

The total area of ③ and ④ can be obtained using integration.

$$\int_0^{t_1} u'(t) dt = u(t_1) - u(0) = u(t_1) = \text{Total value of consumption}$$

To simplify,  
we assume  
 $u(0) = 0$ .

If you simply subtract the value of the rectangle ③ from the integral from 0 to  $t_1$ , you can find the area of ④, the benefit to consumers.

THE BENEFIT FOR THE CONSUMERS ④ IS THE TOTAL VALUE OF CONSUMPTION MINUS THE AMOUNT THEY PAID ③, RIGHT?



YES, THAT'S IT. NOW LET'S LOOK AT THE SUPPLY AND DEMAND CURVES COMBINED TOGETHER.

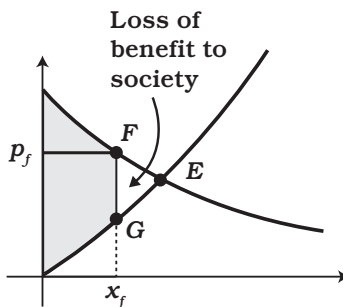
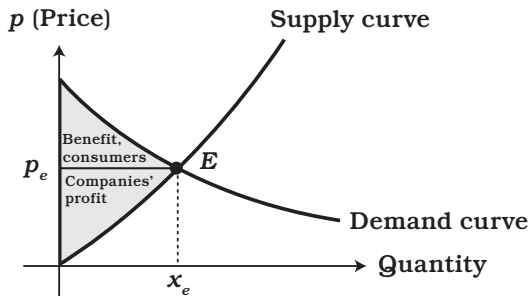


WE CAN SAY THAT THE COMPANIES' PROFIT PLUS THE BENEFIT FOR CONSUMERS EQUALS THE OVERALL BENEFIT FOR SOCIETY, AS ILLUSTRATED BY THE SHADED AREA ON THE RIGHT.

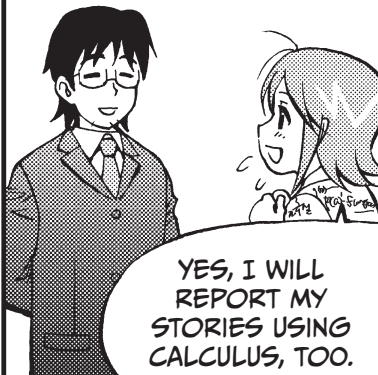


BUT WHAT HAPPENS IF TRADE DOES NOT HAPPEN AT THE PRICE AND QUANTITY DETERMINED BY THE INTERSECTION POINT  $E$ ?

THE OVERALL BENEFIT TO SOCIETY IS REDUCED BY THE AMOUNT CORRESPONDING TO THE EMPTY AREA IN THE FIGURE.

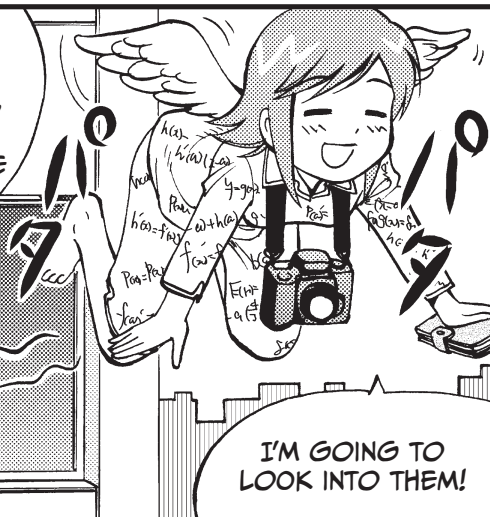


DO YOU GET IT?



YES, I WILL REPORT MY STORIES USING CALCULUS, TOO.

I ALSO THINK VELOCITY AND FALLING BODIES ARE GOOD TOPICS TO WRITE ABOUT.



I'M GOING TO LOOK INTO THEM!

## The Integral of Velocity Proven to Be Distance!

*The integral of velocity = difference in position = distance traveled*

If we understand this formula, it's said that we can correctly calculate the distance traveled for objects whose velocity changes constantly. But is that true? Our promising freshman reporter Noriko Hikima closes in on the truth of this matter in her hard-hitting report.

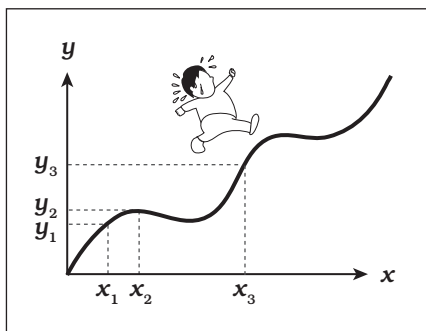


Figure 1: This graph represents Futoshi's distance traveled over time. He moves to point  $y_1$ ,  $y_2$ ,  $y_3$ ... as time progresses to  $x_1$ ,  $x_2$ ,  $x_3$ ...

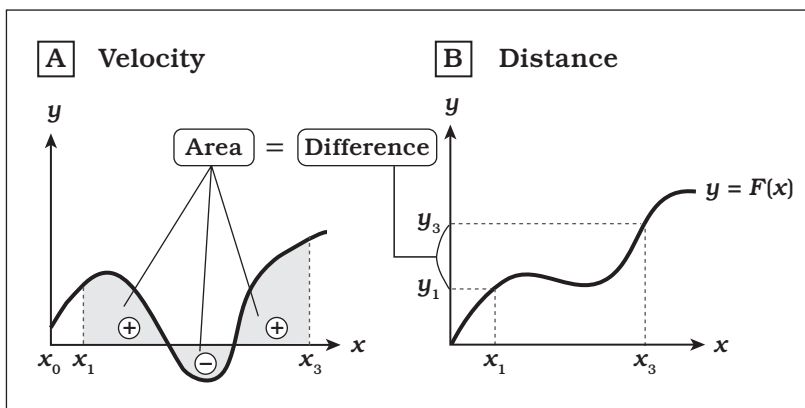


Figure 2

Sanda-Cho—Some readers will recall our earlier example describing Futoshi walking on a moving walkway. Others have likely deliberately blocked his sweaty image from their minds. But you almost certainly remember that the derivative of the distance is the speed.

$$\textcircled{1} \quad y = F(x)$$

$$\textcircled{2} \quad \int_a^b v(x) dx = F(b) - F(a)$$

Equation  $\textcircled{1}$  expresses the position of the monstrous, sweating Futoshi. In other words, after  $x$  seconds he has lumbered a total distance of  $y$ .

*Integral of Velocity = Difference in Position*

The derivative  $F'(x)$  of expression  $\textcircled{1}$  is the “instantaneous velocity” at  $x$  seconds. If we rewrite  $F'(x)$  as  $v(x)$ , using  $v$  for *velocity*, the Fundamental Theorem of Calculus can be used to obtain equation  $\textcircled{2}$ ! Look at the graph of  $v(x)$  in Figure 2-A—Futoshi's velocity over time. The shaded part of the graph is equal to the integral—equation  $\textcircled{2}$ .

But also look at Figure 2-B, which shows the distance Futoshi has traveled over time. If we look at Figures 2-A and 2-B side by side, we see that the integral of the velocity is equal to the difference in position (or distance)! Notice how the two graphs match—

when Futoshi's velocity is positive, his distance increases, and vice versa.

# Free Fall from Tokyo Tower

## How Many Seconds to the Ground?

It's easy to take things for granted—consider gravity. If you drop an object from your hand, it naturally falls to the ground. We can say that this is a motion that changes every second—it is *accelerating* due to the Earth's gravitational pull. This motion can be easily described using calculus.

But let's consider a bigger drop—all the way from the top of Tokyo Tower—and find out, "How many seconds does it take an object to reach the ground?" Pay no attention to Futoshi's remark, "Why don't you go to the top of Tokyo Tower with a stopwatch and find out for yourself?"

The increase in velocity when an object is in free fall is called *gravitational acceleration*, or  $9.8 \text{ m/s}^2$ . In other words, this means that an object's velocity increases by  $9.8 \text{ m/s}$  every second. Why is this the rate of acceleration? Well, let's just assume the scientists are right for today.

Expression ❶ gives the distance the object falls in  $T$  seconds. Since the integral of the velocity is the difference in position (or the distance the object travels), equation ❷ can be derived. Look at Figure 3—we've calculated the area by taking half of the product of the  $x$  and  $y$  values—in this case,  $\frac{1}{2} \times 9.8t \times t$ . And we know that the height of Tokyo Tower is  $333 \text{ m}$ . The square root of  $(333 / 4.9)$  equals about  $8.2$ , so an object takes about  $8.2$  seconds to reach the ground. (We've neglected air resistance here for convenience.)



$$\text{❶ } F(T) - F(0) = \int_0^T v(x) dx = \int_0^T 9.8(x) dx$$

$$\text{❷ } 4.9T^2 - 4.9 \times 0^2 = 4.9T^2$$

$$333 = 4.9T^2 \Rightarrow T = \sqrt{\frac{333}{4.9}} = 8.2 \text{ seconds}$$

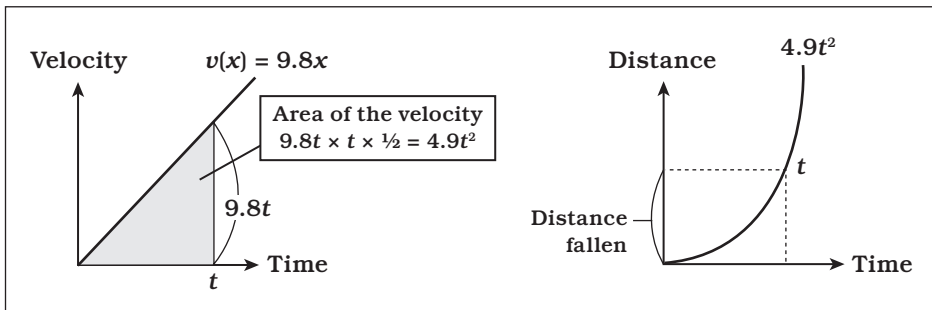


Figure 3

# The Die Is Cast!!!

## The Fundamental Theorem of Calculus Applies to Dice, Too

You probably remember playing games with dice as a child. Since ancient times, these hexahedrons have been rolled around the world, not only in games, but also for fortune telling and gambling.

Mathematically, you can say that dice are the world's smallest random-number generator. Dice are wonderful. Now we'll cast them for calculus! A die can show a 1, 2, 3, 4, 5, or 6—the probability of any one number is 1 in 6. This can be shown with a histogram (Figure 4), with their numbers on the x-axis and the probability on the y-axis.

This can be expressed by equation ❶, or  $f(x) = \text{Probability of rolling } x$ . This becomes equation ❷ when we try to predict a single result—for example, a roll of 4.

- ❶  $f(x) = \text{Probability of rolling } x$
- ❷  $f(4) = \frac{1}{6} = \text{Probability of rolling } 4$

Now let's take a look at Figure 5, which describes a distribution function. First, start at 1 on the x-axis. Since no number less than 1 exists on a die, the probability in this region is 0. At  $x = 1$ , the graph jumps to  $1/6$ , because the probability of rolling a number less than or equal to 1 is 1 in 6. You can also see that the probability of rolling a number equal to or greater than 1 and less than 2 is  $1/6$  as well. This should make intuitive sense. At 2, the probability jumps up to  $2/6$ , which means the probability for rolling a number equal to or less than 2 is  $2/6$ . Since this probability remains until

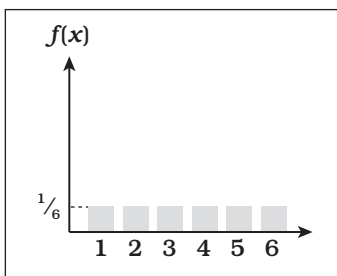


Figure 4: Density function

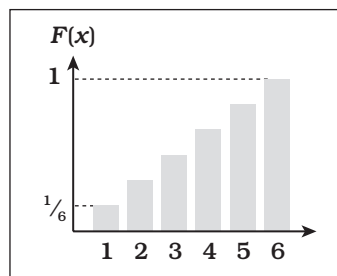


Figure 5: Distribution function

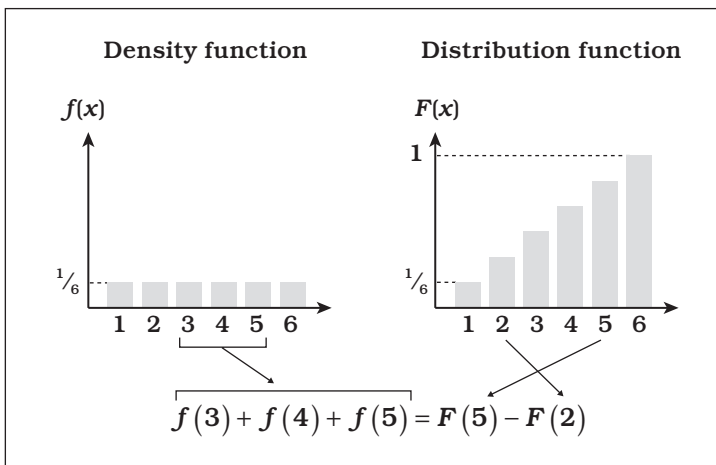


Figure 6: Derivative of distribution function  $F(x)$  = density function  $f(x)$

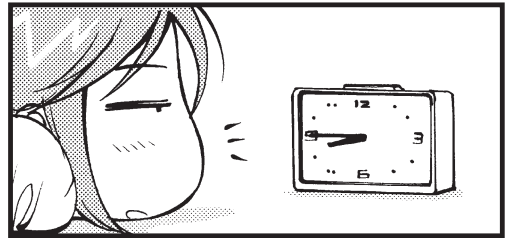
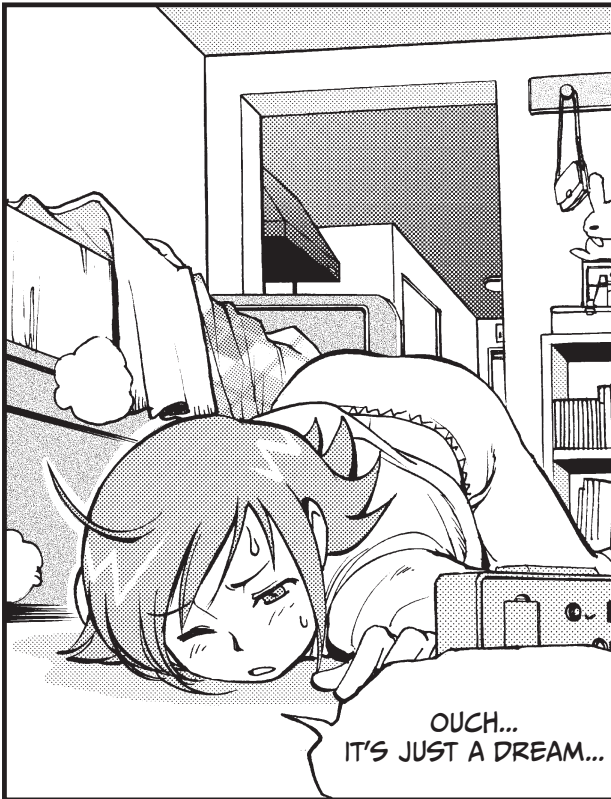
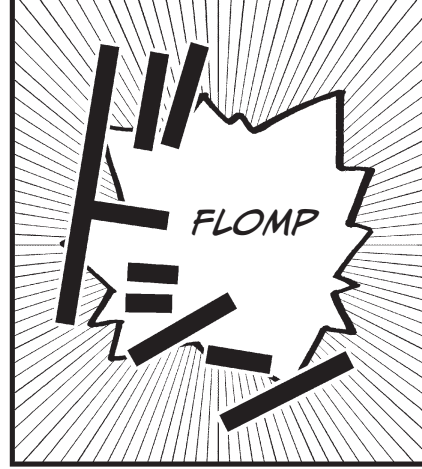
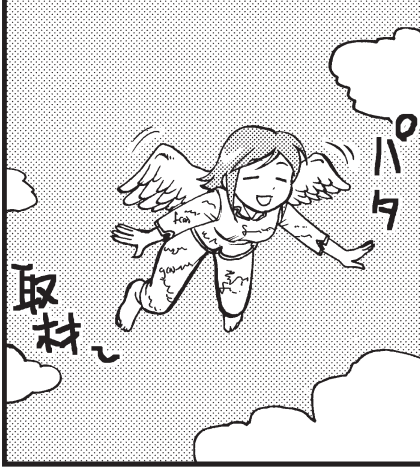
right below 3, the probability of numbers less than 3 is  $2/6$ .

$$\text{❸ } \int_a^b f(x) dx = F(b) - F(a)$$

= Probability of rolling  $x$  where  $a \leq x \leq b$

In the same way, we can find that the probability of rolling a 6 or any number smaller than 6 (that is, any number on the die) is 1. After all, a die cannot stand on one of its corners. Now let's look at the probability of rolling numbers greater than 2 and equal to or less than 5. The equation in Figure 6 explains this relationship.

If we look at equation ❸, we see that it describes what we know—"A definite integral of a differentiated function = The difference in the original function." This is nothing but the Fundamental Theorem of Calculus! How wonderful dice are.



# REVIEW OF THE FUNDAMENTAL THEOREM OF CALCULUS

When the derivative of  $F(x)$  is  $f(x)$ , that is, if  $f(x) = F'(x)$

$$\int_a^b f(x) dx = F(b) - F(a)$$

This can also be written as

$$\int_a^b F'(x) dx = F(b) - F(a)$$

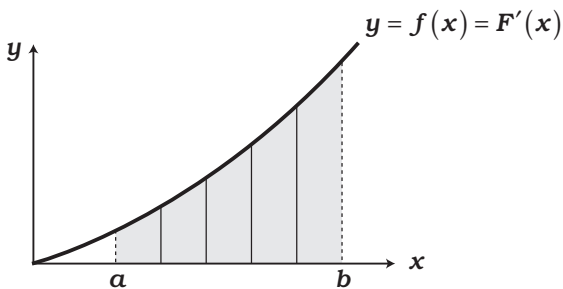


These expressions mean the following.

(Differentiated function)  $dx$   
 = Difference of the original function between  $b$  and  $a$

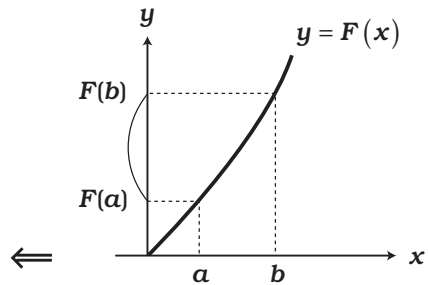
It also means graphically that

$$\left( \begin{array}{l} \text{Area surrounded by the differentiated function} \\ \text{and the x-axis, between } x = a \text{ and } x = b \end{array} \right) = \left( \begin{array}{l} \text{Change in the original} \\ \text{function from } a \text{ to } b \end{array} \right)$$



Fundamental  
Theorem  
of Calculus

$$\int_a^b f(x) dx$$



Difference in the  
original function

## FORMULA OF THE SUBSTITUTION RULE OF INTEGRATION

When a function of  $y$  is substituted for variable  $x$  as  $x = g(y)$ , how do we express

$$S = \int_a^b f(x) dx$$

a definite integral with respect to  $x$ , as a definite integral with respect to  $y$ ?

First, we express the definite integral in terms of a stepwise function approximately as

$$S \approx \sum_{k=0,1,2,\dots,n-1} f(x_k)(x_{k+1} - x_k) \quad (x_0 = a, x_n = b)$$

Transforming variable  $x$  as  $x = g(y)$ , we set

$$y_0 = a, y_1, y_2, \dots, y_n = b$$

so that

$$a = g(y_0), x_1 = g(y_1), x_2 = g(y_2), \dots, b = g(y_n)$$

Note here that using an approximate linear function of

$$x_{k+1} - x_k = g(y_{k+1}) - g(y_k) \approx g'(y_k)(y_{k+1} - y_k)$$

Substituting these expressions in  $S$ , we get

$$S \approx \sum_{k=0,1,2,\dots,n-1} f(x_k)(x_{k+1} - x_k) \approx \sum_{k=0,1,2,\dots,n-1} f(g(y_k))g'(y_k)(y_{k+1} - y_k)$$

The last expression is an approximation of

$$\int_a^b f(g(y))g'(y) dy$$

Therefore, by making the divisions infinitely small, we obtain the following formula.

### FORMULA 3-2: THE SUBSTITUTION RULE OF INTEGRATION

$$\int_a^b f(x) dx = \int_a^b f(g(y))g'(y) dy$$

EXAMPLE:

Calculate:

$$\int_0^1 10(2x + 1)^4 dx$$

We first substitute the variable so that  $y = 2x + 1$ , or  $x = g(y) = \frac{y - 1}{2}$ .

Since  $y = 2x + 1$ , if we take the derivative of both sides, we get  $dy = 2dx$ . Then we get  $dx = \frac{1}{2} dy$ .

Since we now integrate with respect to  $y$ , the new interval of integration is obtained from  $0 = g(1)$  and  $1 = g(3)$  to be  $1 - 3$ .\*

$$\int_0^1 10(2x + 1)^4 dx = \int_1^3 10y^4 \frac{1}{2} dy = \int_1^3 5y^4 dy = 3^5 - 1^5 = 242$$

## THE POWER RULE OF INTEGRATION

In the example above we remembered that  $5y^4$  is the derivative of  $y^5$  to finish the problem. Since we know that if  $F(x) = x^n$ , then  $F'(x) = f(x) = nx^{(n-1)}$ , we should be able to find a general rule for finding  $F(x)$  when  $f(x) = x^n$ .

We know that  $F(x)$  should have  $x^{(n+1)}$  in it, but what about that coefficient? We don't have a coefficient in our derivative, so we'll need to start with one. When we take the derivative, the coefficient will be  $(n + 1)$ , so it follows that  $1 / (n + 1)$  will cancel it out. That means that the general rule for finding the antiderivative  $F(x)$  of  $f(x) = x^n$  is

$$F(x) = \frac{1}{n+1} \times x^{(n+1)} = \frac{x^{(n+1)}}{n+1}$$

---

\* In other words, when  $x = 0$ ,  $y = 1$ , and when  $x = 1$ ,  $y = 3$ . We then use that as the range of our definite integral.

## EXERCISES

1. Calculate the definite integrals given below.

❶  $\int_1^3 3x^2 dx$

❷  $\int_2^4 \frac{x^3 + 1}{x^2} dx$

❸  $\int_0^5 x + (1 + x^2)^7 dx + \int_0^5 x - (1 + x^2)^7 dx$

2. Answer the following questions.

- A. Write an expression of the definite integral which calculates the area surrounded by the graph of  $y = f(x) = x^2 - 3x$  and the x-axis.
- B. Calculate the area given by this expression.

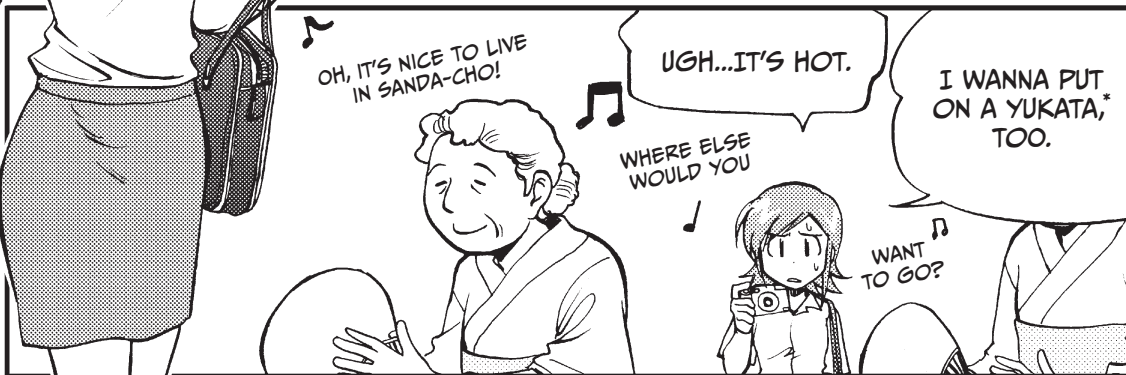
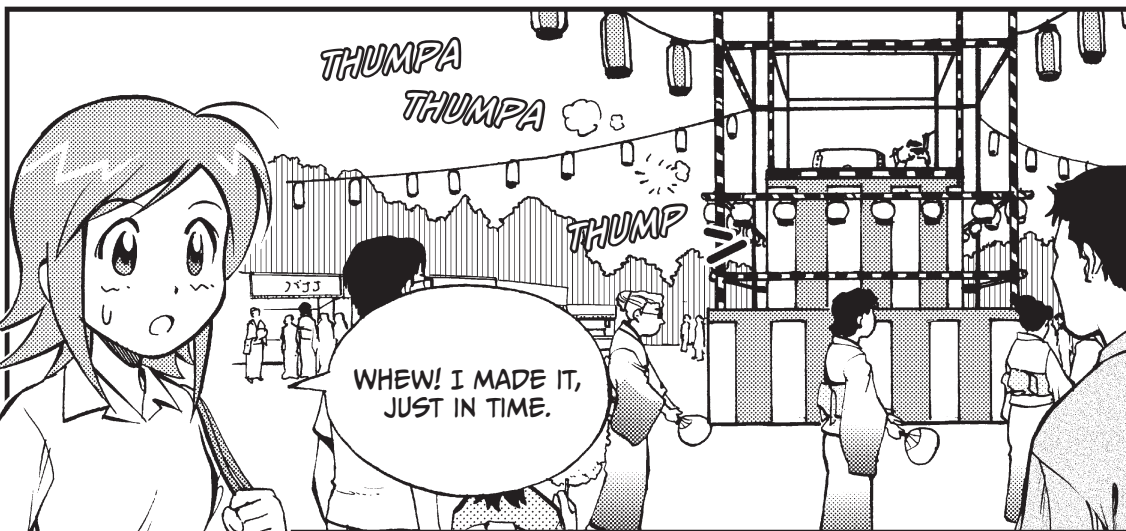


# 4

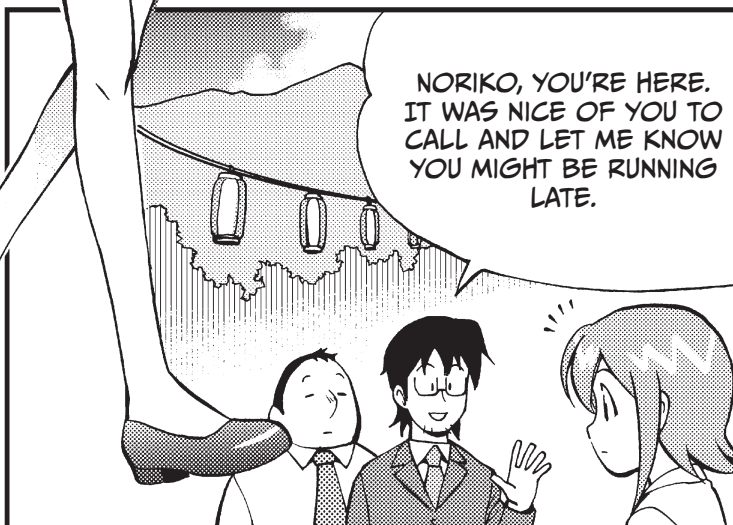
LET'S LEARN INTEGRATION  
TECHNIQUES!

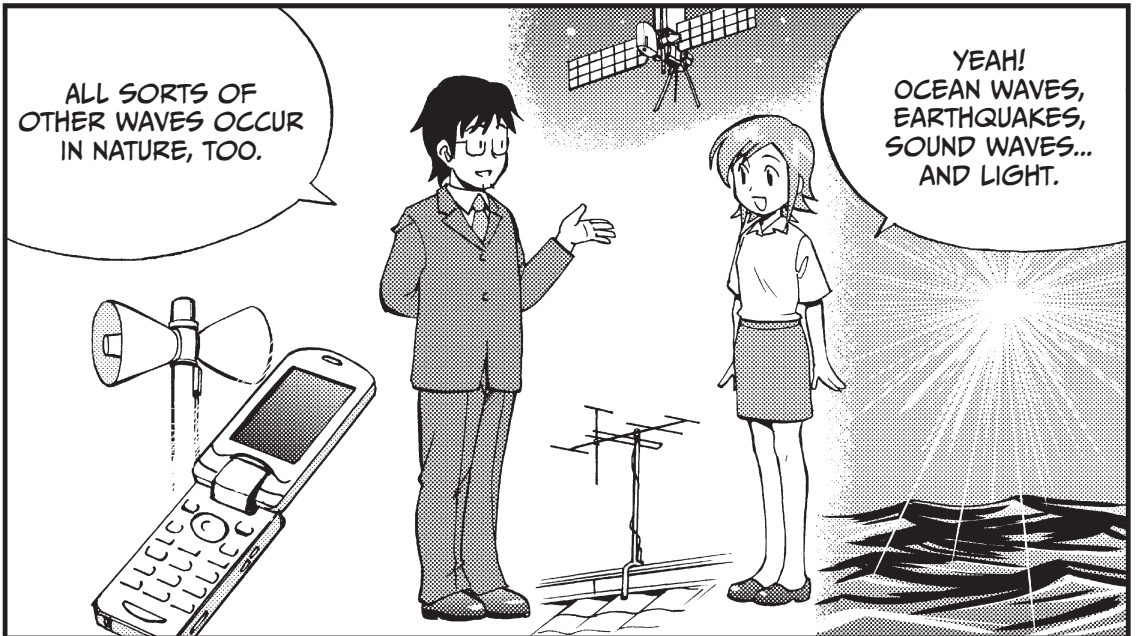
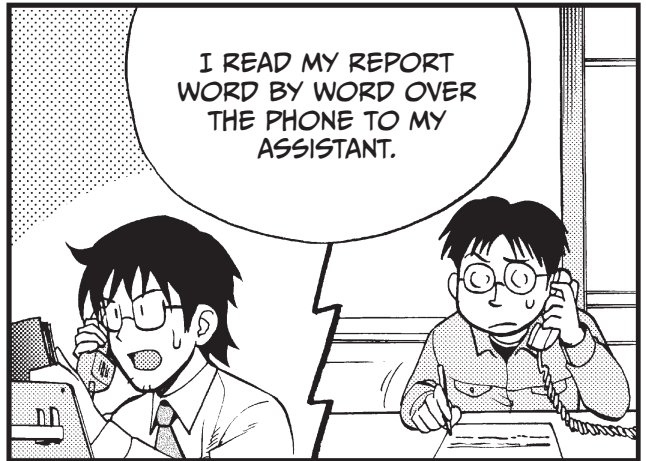
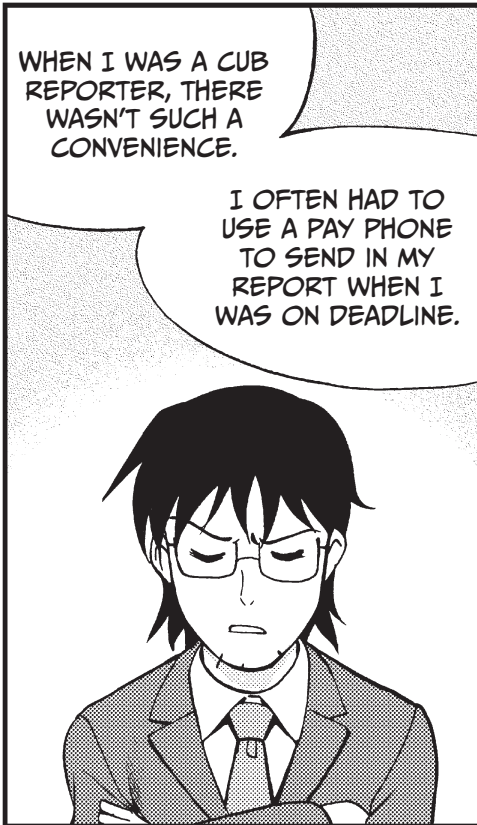


# USING TRIGONOMETRIC FUNCTIONS



\* YUKATA IS TRADITIONAL JAPANESE SUMMER WEAR.





THOSE WAVES CAN BE DESCRIBED WITH FUNCTIONS, FOR EXAMPLE, WITH THE COSINE OF THETA ( $\cos \theta$ ). DID YOU KNOW THAT?

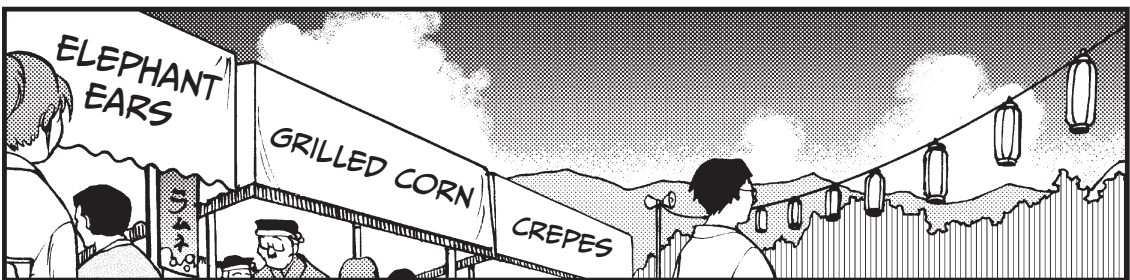
UH, I HAVE TO GO BACK TO WORK.

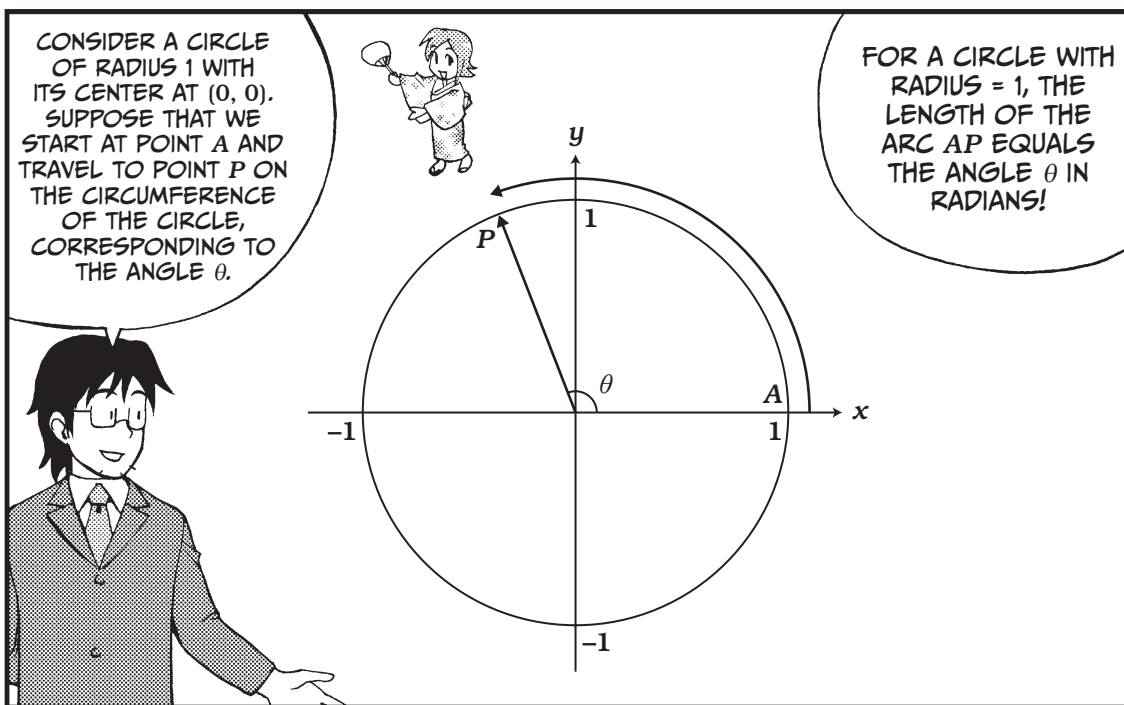
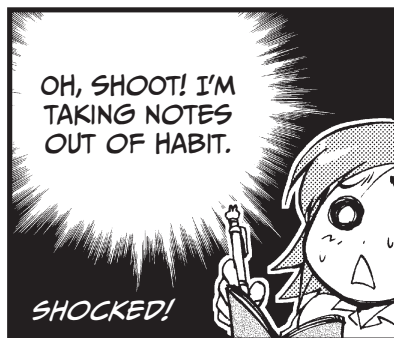
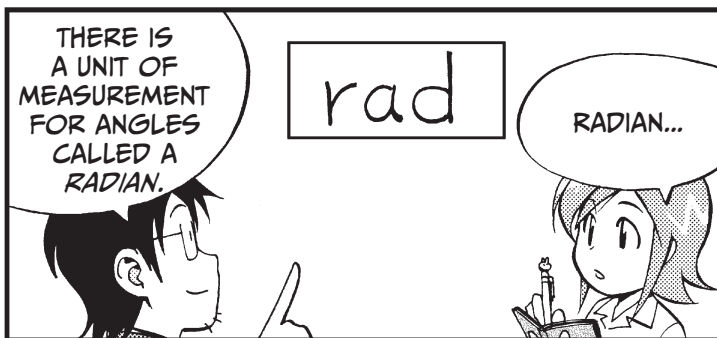
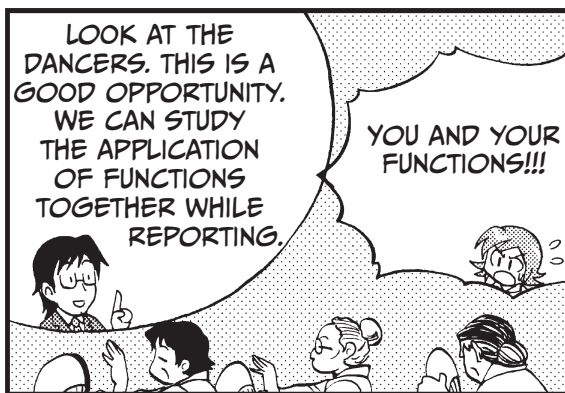
NORIKO!

INCIDENTALLY, IF YOU CUT OUT A SLEEVE OF A BLOUSE, THE CUT END IS A GRAPH OF  $\cos \theta$ .

TRIGONOMETRIC FUNCTIONS ARE VERY IMPORTANT FOR FASHION!

GEEZ.





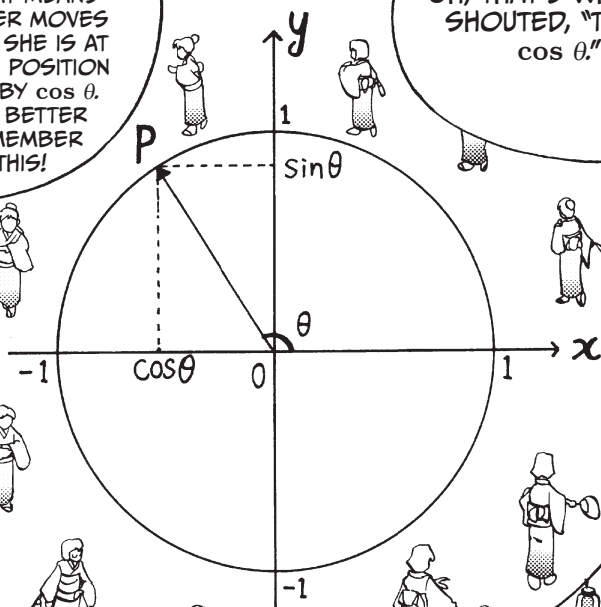
BECAUSE THE TOTAL CIRCUMFERENCE OF THIS CIRCLE IS  $2\pi$ , WE KNOW THAT 90 DEGREES =  $\frac{\pi}{2}$  RADIANS AND 180 DEGREES =  $\pi$  RADIANS. A RADIAN IS ABOUT EQUAL TO 57.2958 DEGREES.

FROM NOW ON, WE WILL USE RADIANS AS THE UNIT FOR ANY ANGLE.

AND WE CAN EXPRESS  $x$  AS THE FUNCTION  $\cos \theta = x$ . THAT MEANS WHEN A DANCER MOVES BY  $\theta$  RADIANS, SHE IS AT A HORIZONTAL POSITION DETERMINED BY  $\cos \theta$ . YOU BETTER REMEMBER THIS!

OH, THAT'S WHY YOU SHOUTED, "THAT'S  $\cos \theta$ ."

WHAT'S GOING ON INSIDE HIS HEAD?

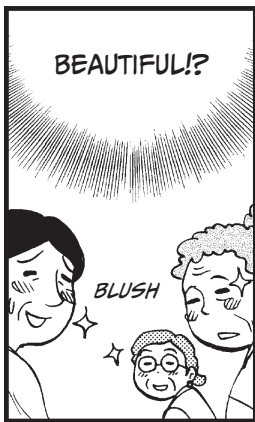
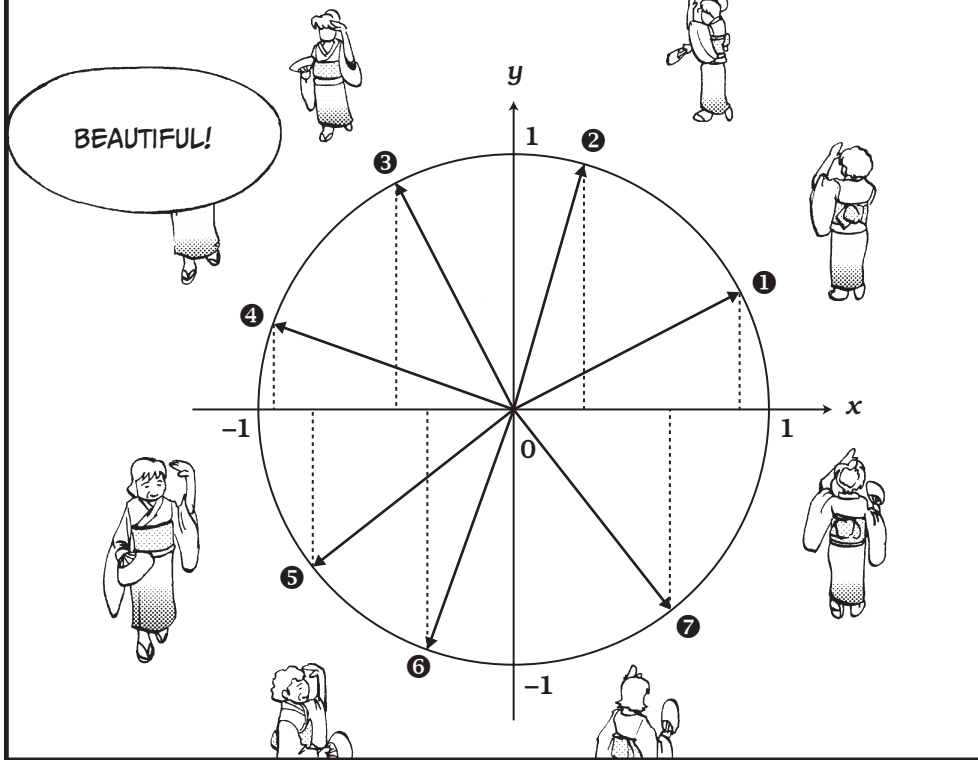


IN THE SAME WAY, THE DANCER'S VERTICAL POSITION CAN BE EXPRESSED AS THE FUNCTION  $\sin \theta = y$ .

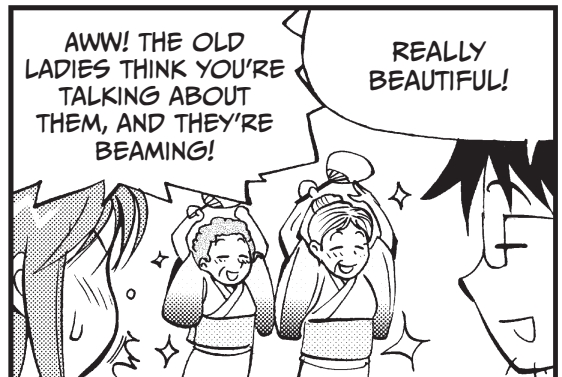
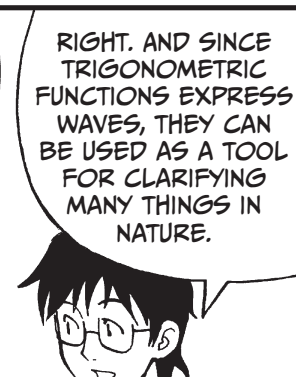
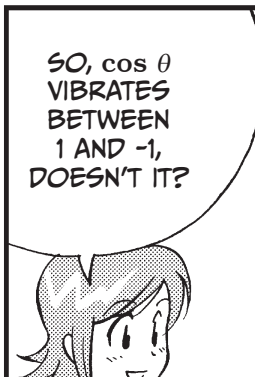
UM, OKAY...

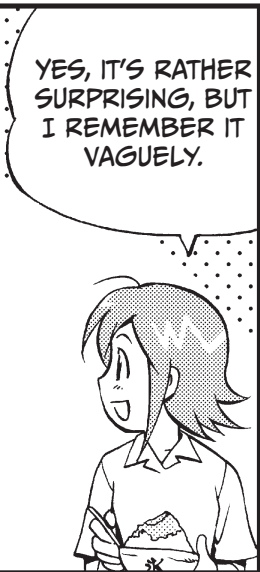
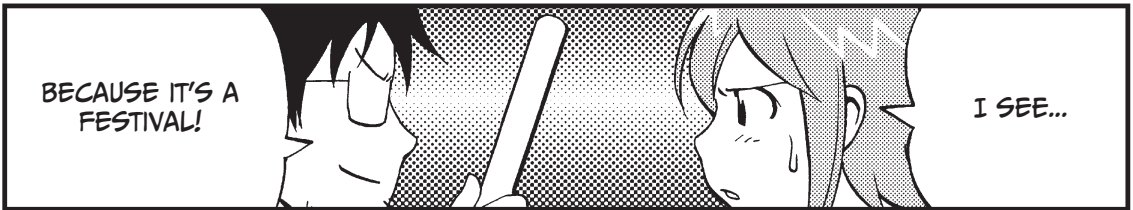
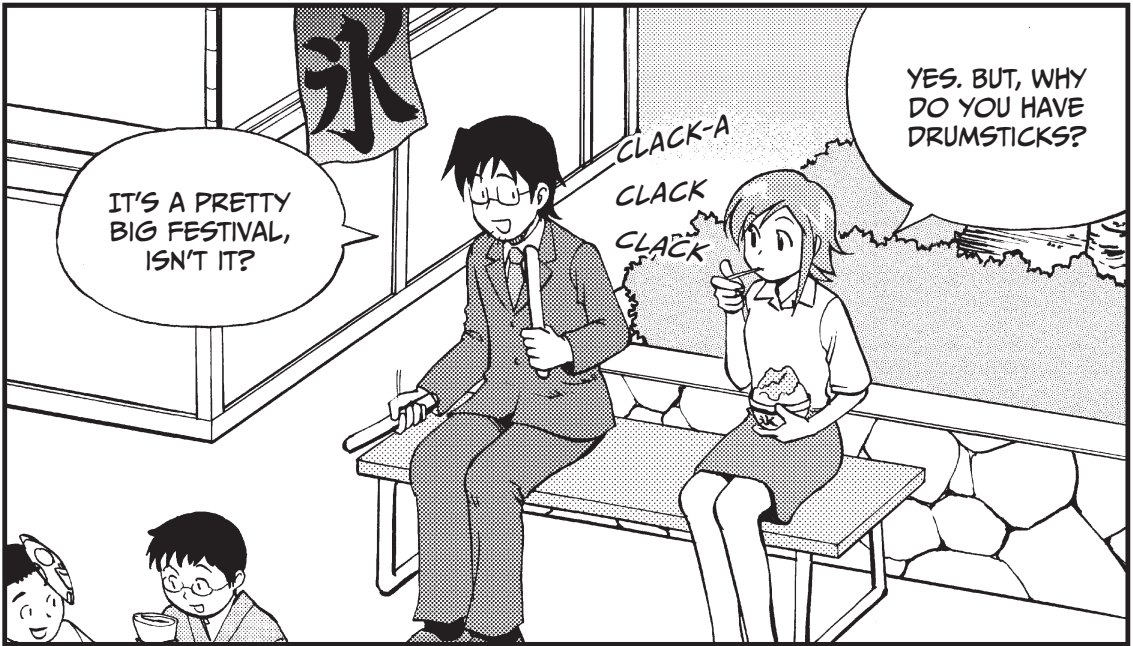
LOOK, NORIKO!!

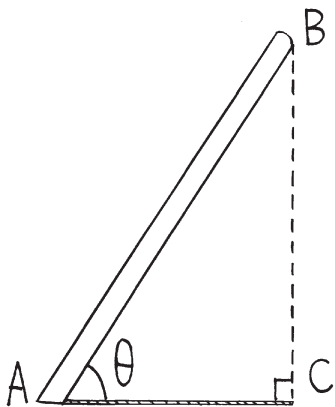
YANK!



YES! AS  $\theta$  BECOMES LARGER, THE VALUE OF  $\cos \theta$  CHANGES FROM 1, GRADUALLY BECOMES SMALLER UNTIL IT'S 0, GOES ALL THE WAY DOWN TO  $-1$ , BACK TO 0, THEN BACK TO 1 AGAIN!

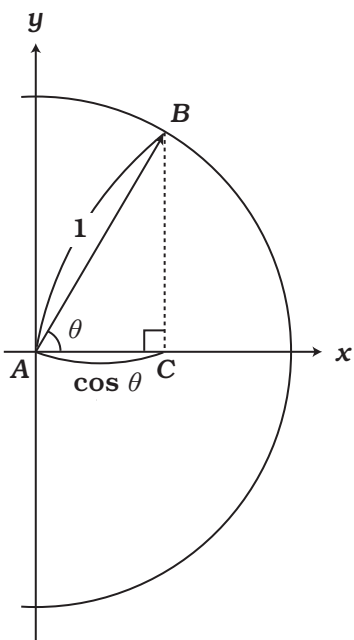






THE SUN IS SHINING STRAIGHT DOWN ON STICK AB, WHICH IS STANDING TILTED AT ANGLE  $\theta$  FROM THE GROUND.

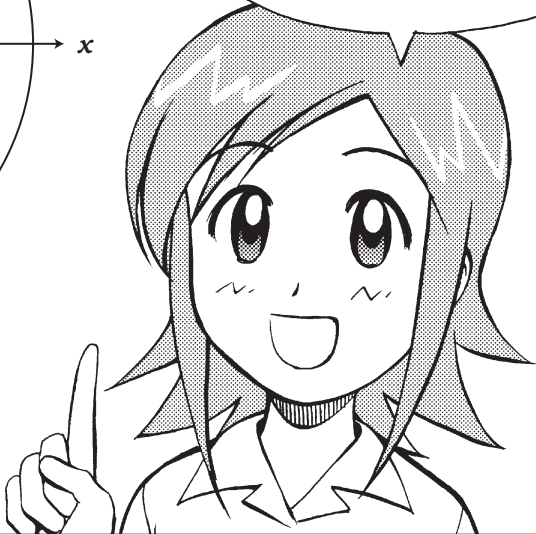
IF WE ASSUME THE RESULTING SHADOW (THE ORTHOGONAL PROJECTION) TO BE AC, THE LENGTH OF SHADOW AC EQUALS THE LENGTH OF STICK AB MULTIPLIED BY  $\cos \theta$ .



WE CAN THINK OF THE STICK IN TERMS OF A FUNCTION. AND BY DEFINITION,

$$\cos \theta = \frac{AC \text{ (shadow)}}{AB \text{ (stick)}}$$

SO THE SHADOW'S LENGTH IS  $AB \times \cos \theta$ . RIGHT?



THAT'S RIGHT! COSINE EXPRESSES HOW MUCH SHORTER THE SHADOW IS THAN THE STICK ITSELF!



INCIDENTALLY, SINCE THE X-AXIS COINCIDES WITH THE Y-AXIS WHEN IT IS ROTATED BY 90 DEGREES ( $\frac{\pi}{2}$  RADIANS), WE CAN SAY  $\sin \theta$  IS A FUNCTION THAT OUTPUTS, DELAYED BY  $\frac{\pi}{2}$ , THE SAME VALUES AS  $\cos \theta$ .

IN OTHER WORDS,

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$$

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta$$

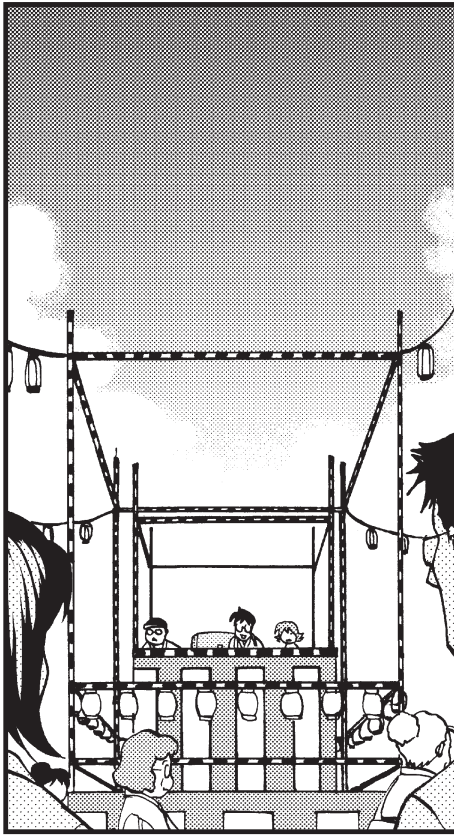
YES?

UH...WILL YOU GIVE US BACK OUR DRUMSTICKS?

NOW, WE ARE READY FOR THE MAIN PART OF THE SANDA SUMMER FESTIVAL!!

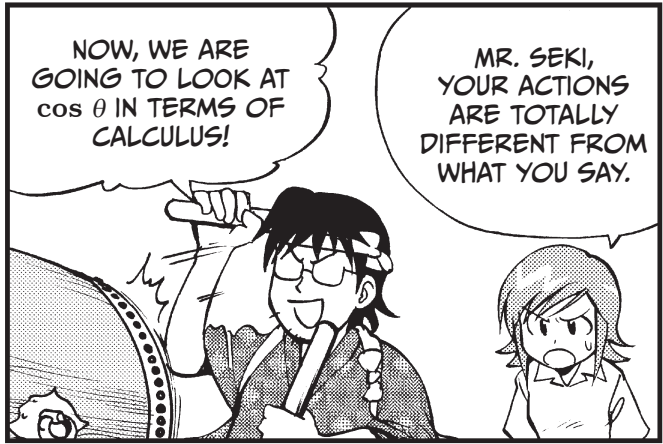
OOPS!

# USING INTEGRALS WITH TRIGONOMETRIC FUNCTIONS



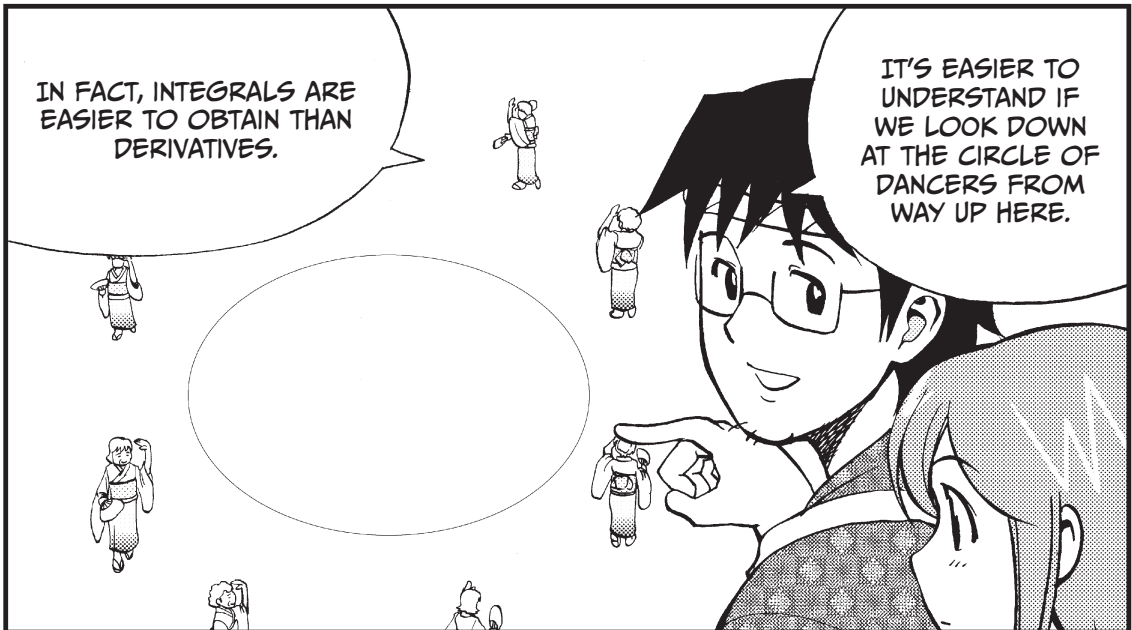
HERE ARE SPECIAL SEATS FOR YOU. BE CAREFUL NOT TO FALL, REPORTERS, AND TAKE GOOD PICTURES.

OKAY. WE WILL.



NOW, WE ARE GOING TO LOOK AT  $\cos \theta$  IN TERMS OF CALCULUS!

MR. SEKI, YOUR ACTIONS ARE TOTALLY DIFFERENT FROM WHAT YOU SAY.

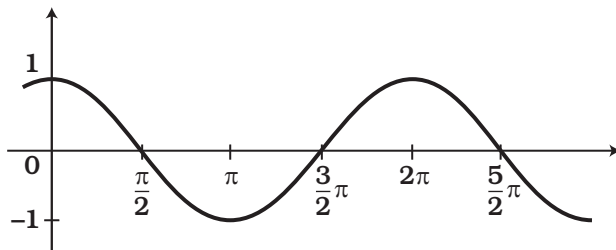


IN FACT, INTEGRALS ARE EASIER TO OBTAIN THAN DERIVATIVES.

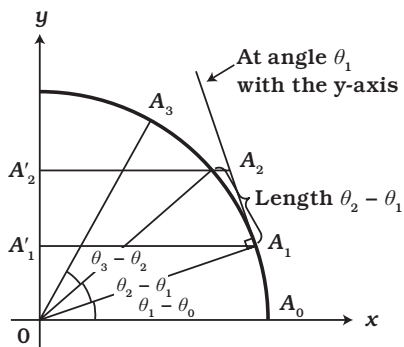
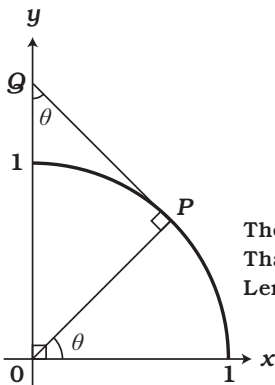
IT'S EASIER TO UNDERSTAND IF WE LOOK DOWN AT THE CIRCLE OF DANCERS FROM WAY UP HERE.

WHAT WE NEED TO DO IS TO FIND OUT WHAT  
 $\sum \cos \theta \times \Delta \theta = \cos \theta_0 (\theta_1 - \theta_0) + \cos \theta_1 (\theta_2 - \theta_1) + \dots$   
 $+ \cos \theta_{n-1} (\theta_n - \theta_{n-1})$  BECOMES.

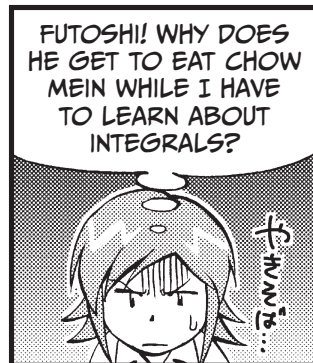
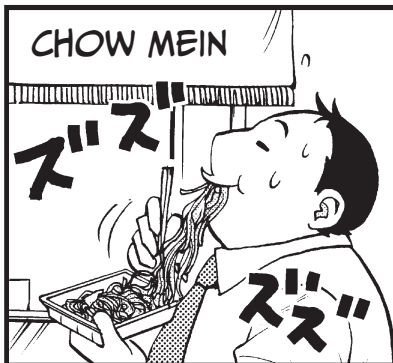
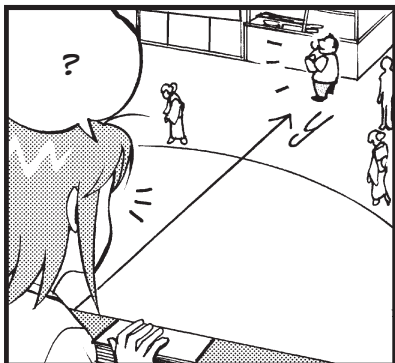
LOOKING AT THIS  
 PUTS ME IN A  
 FOG.

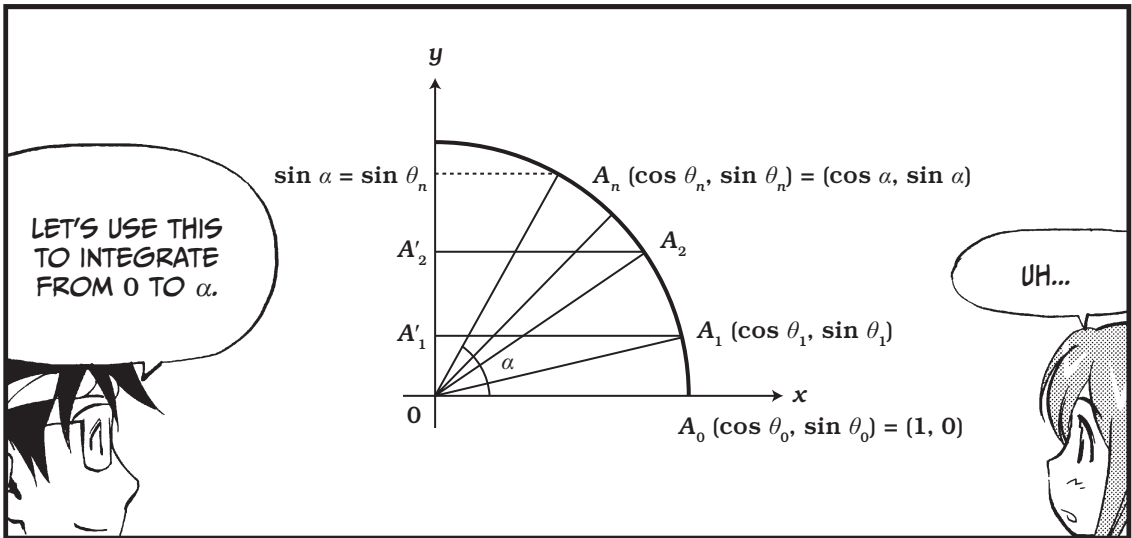


LOOK AT THIS  
 FIGURE. DOESN'T  
 THIS GIVE YOU A  
 GOOD IDEA? THIS  
 SHOWS THAT THE  
 INTERSECTING  
 ANGLE OF THE  
 Y-AXIS WITH THE  
 TANGENT LINE  $PQ$ ,  
 WHERE  $P$  IS THE  
 POINT MOVED FROM  
 $(1, 0)$  BY ANGLE  $\theta$ ,  
 IS ALSO  $\theta$ .



The change in  $\cos \theta$  is the length  $A'_1A'_2$ .  
 That length is the orthogonal projection  $A_1A_2$ .  
 Length  $A'_1A'_2 \approx \text{arc } A_1A_2 \times \cos \theta_1 = (\theta_2 - \theta_1) \times \cos \theta_1$





$\sum \cos \theta \Delta \theta$  when  $\theta$  is changed from 0 to  $\alpha$

$\cos \theta_0 (\theta_1 - \theta_0) + \cos \theta_1 (\theta_2 - \theta_1) + \dots + \cos \theta_{n-1} (\theta_n - \theta_{n-1})$

$\approx A'_0 A'_1 + A'_1 A'_2 + \dots + A'_{n-1} A'_n = A'_0 A'_n = \sin \alpha$

IS THIS RIGHT?

RIGHT! IF WE MAKE THESE INFINITELY SMALL...

WE FIND THAT THE INTEGRAL OF COSINE IS SINE.

$\int_0^\alpha \cos \theta d\theta = \sin \alpha - \sin 0$

THEN, TO PUT IT THE OTHER WAY AROUND, THE DERIVATIVE OF SINE IS COSINE?

YOU'RE RIGHT!

NOW, REMEMBER THESE FORMULAS.

FORMULA 4-1: THE DIFFERENTIATION AND INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Since ①  $\int_0^{\alpha} \cos \theta d\theta = \sin \alpha - \sin 0$ , we know that sine must be cosine's derivative.

②  $(\sin \theta)' = \cos \theta$

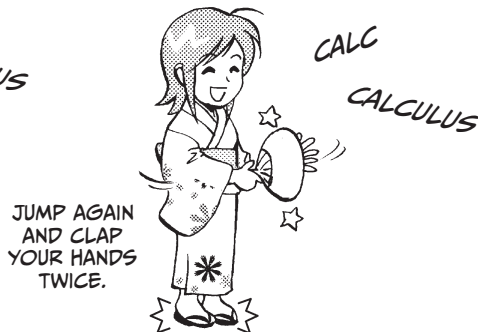
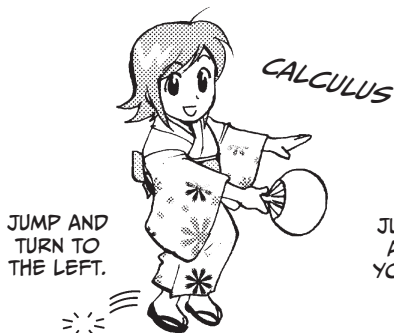
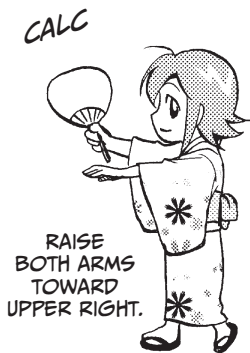
Now, substitute  $\theta + \frac{\pi}{2}$  for  $\theta$  in ②. We get  $\left\{ \sin \left( \theta + \frac{\pi}{2} \right) \right\}' = \cos \left( \theta + \frac{\pi}{2} \right)$ .  
Using the equations from page 124,  
we then know that

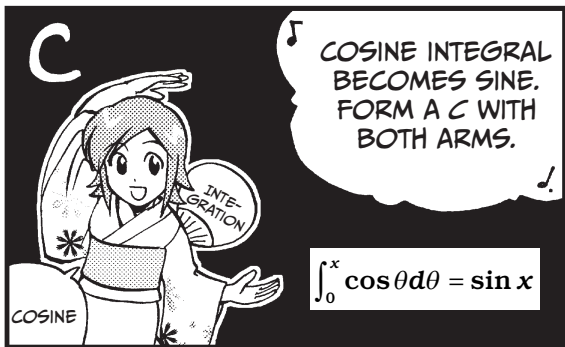
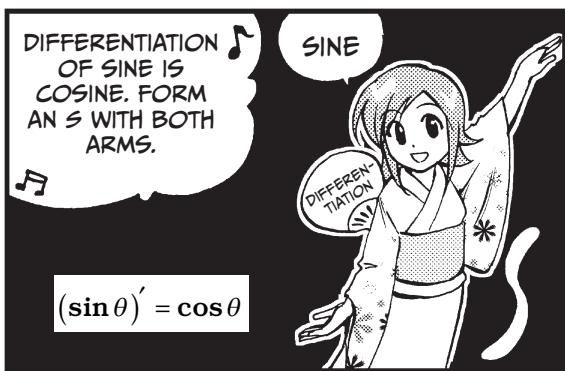
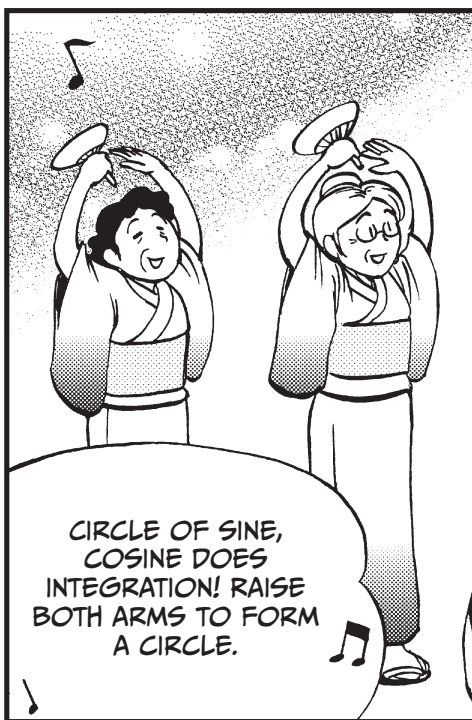
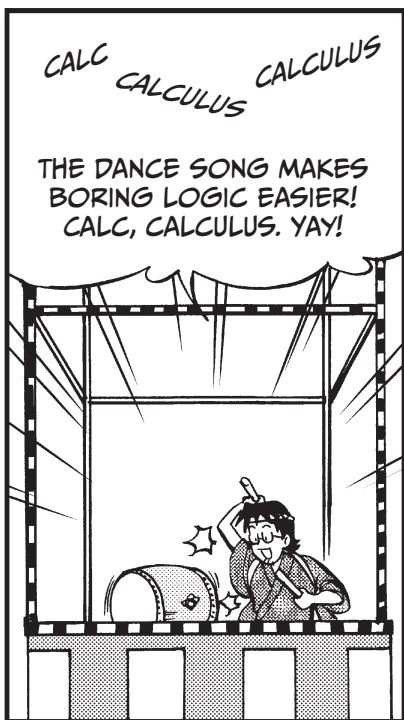
③  $(\cos \theta)' = -\sin \theta$

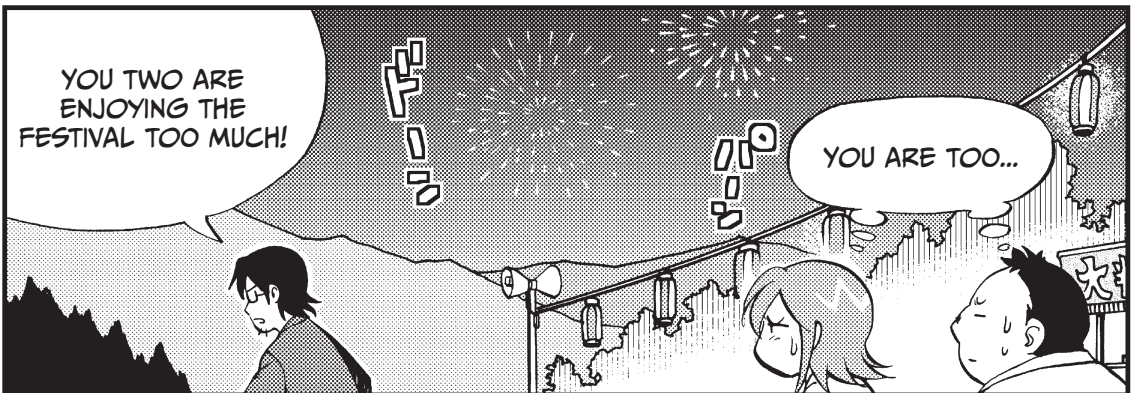
We find that differentiating or integrating sine gives cosine and vice versa.



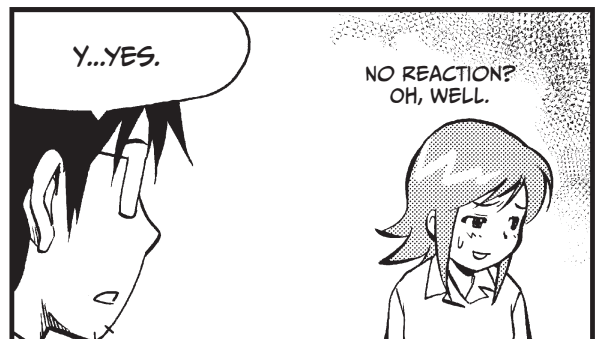
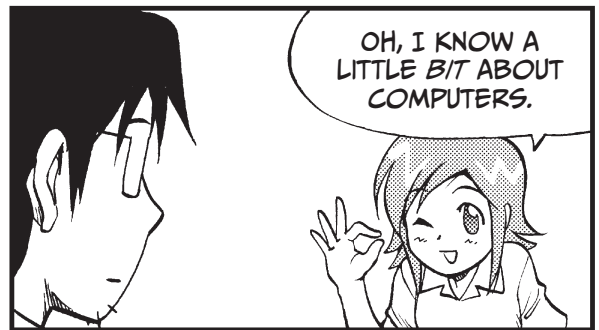
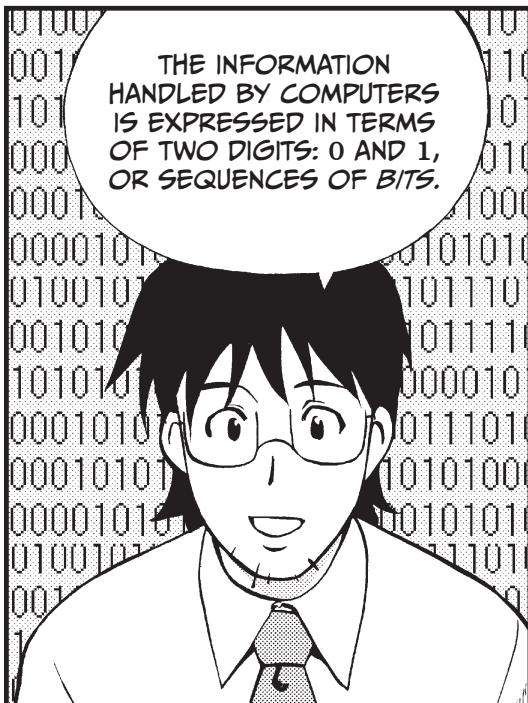
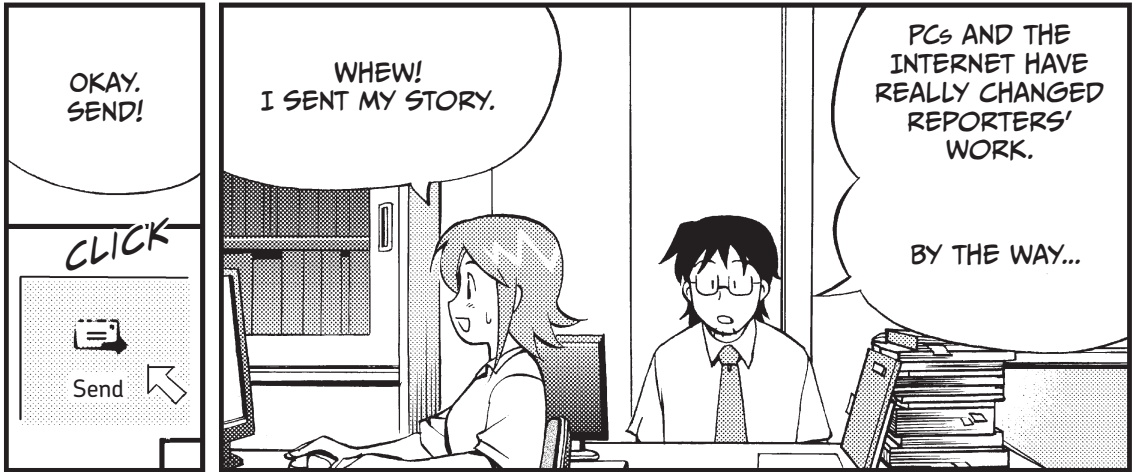
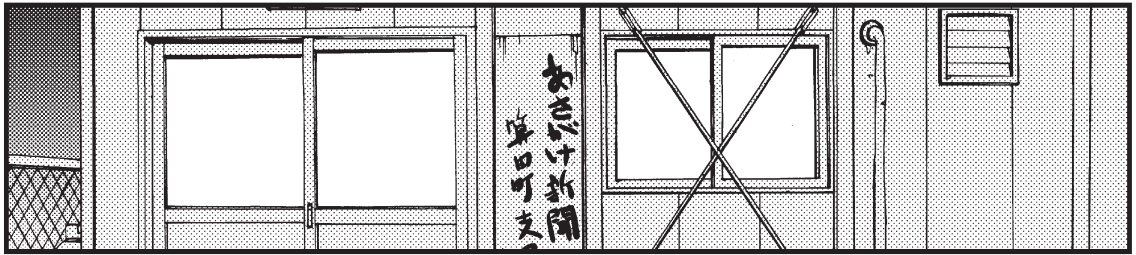
**CALCULUS DANCE SONG**  
TRIGONOMETRIC VERSION







# USING EXPONENTIAL AND LOGARITHMIC FUNCTIONS



SINCE COMPUTERS HANDLE INFORMATION IN THE BINARY SYSTEM, ONE BIT CAN REPRESENT TWO NUMBERS (0 AND 1); TWO BITS CAN REPRESENT FOUR (00, 01, 10, AND 11); THREE BITS CAN REPRESENT EIGHT; AND  $n$  BITS CORRESPOND TO  $2^n$  POSSIBLE NUMBERS.

IF WE SUPPOSE  $f(x)$  IS THE NUMBER OF VALUES THAT CAN BE EXPRESSED BY  $x$  BITS, THEN  $f(x) = 2^x$ , WHICH IS AN EXPONENTIAL FUNCTION.

EXPONENTIAL FUNCTION

EXPONENTIAL FUNCTION?

AN EXPONENTIAL FUNCTION CAN EXPRESS AN INCREASE LIKE ECONOMIC GROWTH.

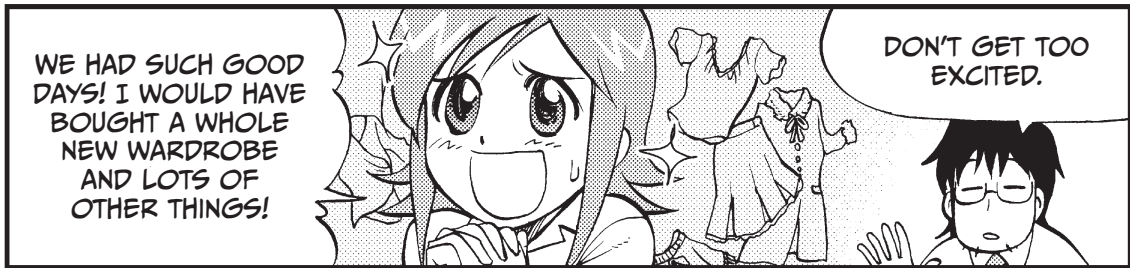
LET ME SEE... FOR EXAMPLE...

WELL...

IN THE 1950S IN JAPAN, WE HAD A HIGH RATE OF ECONOMIC GROWTH: ABOUT 10 PERCENT A YEAR.

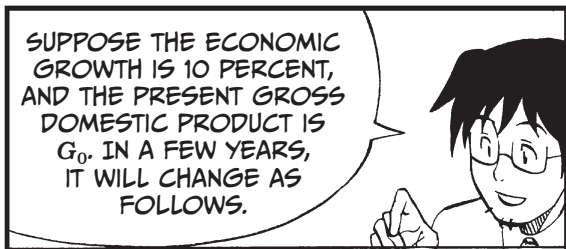
A PERSON WITH AN ANNUAL INCOME OF ¥5 MILLION ONE YEAR EARNED ¥5.5 MILLION THE NEXT YEAR.

HIS SALARY INCREASED 10 PERCENT, AND HE COULD ENJOY 10 PERCENT MORE COMMODITIES AND SERVICES THAN IN THE PREVIOUS YEAR.

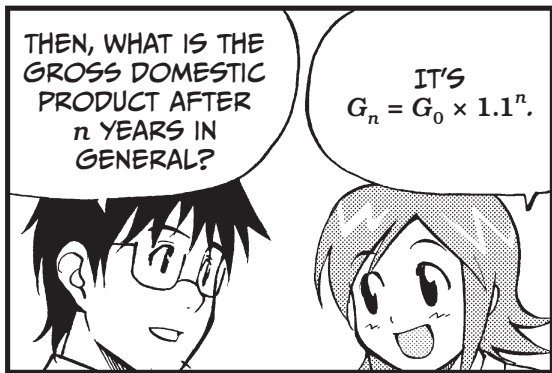


WE HAD SUCH GOOD DAYS! I WOULD HAVE BOUGHT A WHOLE NEW WARDROBE AND LOTS OF OTHER THINGS!

DON'T GET TOO EXCITED.



SUPPOSE THE ECONOMIC GROWTH IS 10 PERCENT, AND THE PRESENT GROSS DOMESTIC PRODUCT IS  $G_0$ . IN A FEW YEARS, IT WILL CHANGE AS FOLLOWS.



THEN, WHAT IS THE GROSS DOMESTIC PRODUCT AFTER  $n$  YEARS IN GENERAL?

IT'S  $G_n = G_0 \times 1.1^n$ .

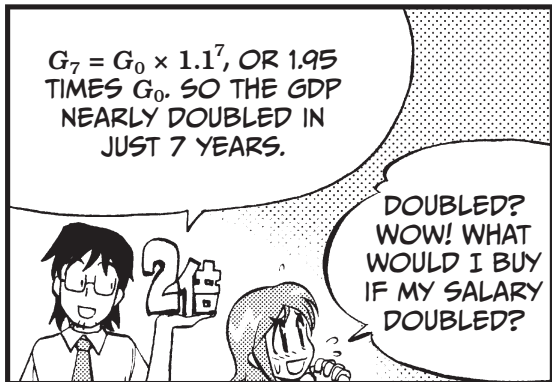
$G_1 = G_0 \times 1.1$   
Gross domestic product after 1 year

$G_2 = G_1 \times 1.1 = G_0 \times 1.1^2$   
Gross domestic product after 2 years

$G_3 = G_0 \times 1.1^3$   
Gross domestic product after 3 years

$G_4 = G_0 \times 1.1^4$   
Gross domestic product after 4 years

$G_5 = G_0 \times 1.1^5$   
Gross domestic product after 5 years



$G_7 = G_0 \times 1.1^7$ , OR 1.95 TIMES  $G_0$ . SO THE GDP NEARLY DOUBLED IN JUST 7 YEARS.

DOUBLED? WOW! WHAT WOULD I BUY IF MY SALARY DOUBLED?



SO, A FUNCTION IN A FORM LIKE  $f(x) = a_0 a^x$  IS CALLED AN EXPONENTIAL FUNCTION.

AN ECONOMY HAVING AN ANNUAL GROWTH RATE OF  $\alpha$  IS EXPRESSED WITH THE EXPONENTIAL FUNCTION  $f(x) = a_0(1 + \alpha)^x$ .

I JUST TOLD YOU THAT BITS ARE CODES FOR EXPRESSING INFORMATION.

YES, 1 BIT IS FOR 2 PATTERNS, 2 BITS FOR 4 PATTERNS.

INVERSE FUNCTION

BITS ARE ALSO AN EXPONENTIAL FUNCTION. IF  $x$  BITS CORRESPOND TO  $f(x)$  POSSIBLE NUMBERS, THEN  $f(x) = 2^x$ . YOU KNOW, THERE IS A FUNCTION CALLED AN INVERSE FUNCTION, WHICH TURNS WHAT YOU CALLED PATTERNS BACK INTO BITS.

IT'S EASY—YOU JUST NEED TO THINK THE OTHER WAY AROUND.

2 PATTERNS → 1 BIT  
 4 PATTERNS → 2 BITS  
 8 PATTERNS → 3 BITS  
 ⋮

SO, WE CAN REPRESENT  $2^n$  POSSIBLE NUMBERS USING  $n$  BITS.

NOW, ASSUME  $g(y)$  IS THE INVERSE FUNCTION OF  $f(x)$ , WHICH TURNS  $y$  PATTERNS BACK INTO BITS. TRY IT.

WE GET  $g(2) = 1$ ,  
 $g(4) = 2$ ,  $g(8) = 3$ ,  
 $g(16) = 4...$

SO, THE RELATIONSHIP BETWEEN  $f$  AND  $g$  CAN BE EXPRESSED AS  $g(f(x)) = x$  AND  $f(g(y)) = y$ .

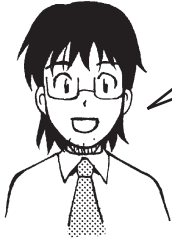
REMEMBER NOW THAT THE INVERSE FUNCTION OF AN EXPONENTIAL FUNCTION IS CALLED A LOGARITHMIC FUNCTION AND IS EXPRESSED WITH THE SYMBOL  $\log$ .

**log**

IN THE ABOVE CASE, IT IS EXPRESSED AS  $g(y) = \log_2 y$ .

RIGHT, AND  $\log_2 2 = 1$ ,  
 $\log_2 4 = 2$ ,  $\log_2 8 = 3$ ,  
 $\log_2 16 = 4...$

## GENERALIZING EXPONENTIAL AND LOGARITHMIC FUNCTIONS



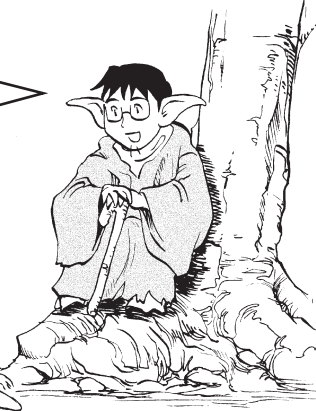
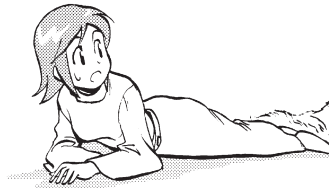
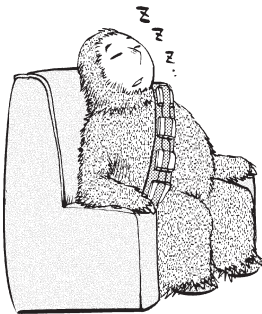
ALTHOUGH EXPONENTIAL AND LOGARITHMIC FUNCTIONS ARE CONVENIENT, OUR DEFINITION OF THEM UP TO NOW ALLOWS ONLY NATURAL NUMBERS FOR  $x$  IN  $f(x) = 2^x$  AND THE POWERS OF 2 FOR  $y$  IN  $g(y) = \log_2 y$ . WE DON'T HAVE A DEFINITION FOR THE  $-8$ th POWER, THE  $7/3$ rd POWER OR THE  $\sqrt{2}$ th POWER,  $\log_2 5$ , OR  $\log_2 \pi$ .

I WILL TELL YOU HOW WE DEFINE EXPONENTIAL AND LOGARITHMIC FUNCTIONS IN GENERAL, USING EXAMPLES.

HMM, WHAT DO WE DO, THEN?



GLAD THAT YOU ASKED AM I. THE POWER OF CALCULUS WE USE FOR THIS. YES.



FIRST, USING OUR EARLIER EXAMPLE, LET'S CHANGE THE ECONOMY'S ANNUAL GROWTH RATE TO ITS INSTANTANEOUS GROWTH RATE.

$$\text{Annual growth rate} = \frac{\text{Value after 1 year} - \text{Present value}}{\text{Present value}} = \frac{f(x+1) - f(x)}{f(x)}$$



THIS IS THE EXPRESSION WE START WITH.



NOW WE DEVELOP THIS INTO THE INSTANTANEOUS GROWTH RATE, AS FOLLOWS.

**Instantaneous growth rate**

= Idealization of  $\left( \frac{\text{Value slightly later} - \text{Present value}}{\text{Present value}} \div \text{Time elapsed} \right)$

= Result obtained by letting  $\varepsilon \rightarrow 0$  in  $\left( \frac{f(x+\varepsilon) - f(x)}{f(x)} \right) \frac{1}{\varepsilon}$

=  $\lim_{\varepsilon \rightarrow 0} \frac{1}{f(x)} \left( \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \right) = \frac{1}{f(x)} f'(x)$



SO, WE DEFINE THE INSTANTANEOUS GROWTH RATE AS  $\frac{f'(x)}{f(x)}$

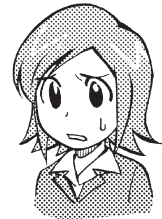
Now, let's consider a function that satisfies the instantaneous growth rate when it is constant, or

$\frac{f'(x)}{f(x)} = c$  where  $c$  is a constant.

Here we assume  $c = 1$ , and we will find  $f(x)$  that satisfies

$\frac{f'(x)}{f(x)} = 1$

FIND  $f(x)$ ? BUT HOW DO WE FIND IT?



1. We first guess this is an exponential function.

SINCE  $f'(x) = f(x)$ ,  $\bullet$   $f'(0) = f(0)$   
NOW, RECALL THAT WHEN  $h$  WAS CLOSE ENOUGH TO ZERO,  
WE HAD  $f(h) \approx f'(0)(h - 0) + f(0)$



From ❶, we have  $f(h) \approx f(0)h + f(0)$  and get

$$\text{❷} \quad f(h) \approx f(0)(h + 1)$$

If  $x$  is close enough to  $h$ , we have

$$f(x) \approx f'(h)(x - h) + f(h)$$

Replacing  $x$  with  $2h$  and using  $f'(h) = f(h)$ ,

$$f(2h) \approx f'(h)(2h - h) + f(h)$$

$$f(2h) \approx f(h)(h) + f(h)$$

$$f(2h) \approx f(h)(h + 1)$$

We'll then substitute  $f(h) \approx f(0)(h + 1)$  into our equation.

$$f(2h) \approx f(0)(h + 1)(h + 1)$$

$$f(2h) \approx f(0)(h + 1)^2$$

In the same way, we substitute  $3h, 4h, 5h, \dots$ , for  $x$  and allow  $mh = 1$ .

$$f(1) = f(mh) \approx f(0)(h + 1)^m$$

Similarly,

$$f(2) = f(2mh) \approx f(0)(h + 1)^{2m} = f(0)\{(1 + h)^m\}^2$$

$$f(3) = f(3mh) \approx f(0)(h + 1)^{3m} = f(0)\{(1 + h)^m\}^3$$

Thus, we get

$$f(n) \approx f(0)a^n \quad \text{where we used } a = (1 + h)^m$$

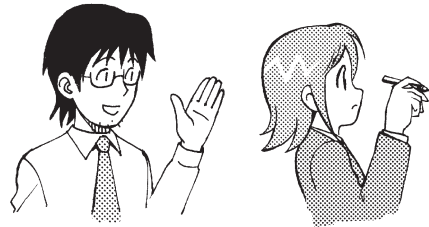
which is suggestive of an exponential function.\*

---

\* Since  $mh = 1$ ,  $h = \frac{1}{m}$ . Then,  $f(1) \approx f(0)\left(1 + \frac{1}{m}\right)^m$ . If we let  $m \rightarrow \infty$  here,  $\left(1 + \frac{1}{m}\right)^m \rightarrow e$ , or *Euler's number*, a number about equal to 2.718. Thus,  $f(1) = f(0) \times e$ , which is consistent with the discussion on page 141.

2. Next we will find out that  $f(x)$  surely exists and what it is like.

EXPRESS THE INVERSE FUNCTION  
OF  $y = f(x)$  AS  $x = g(y)$ .



FROM  $f'(x) = f(x)$  INDICATED ON PAGE 136, THE DERIVATIVE  
OF  $f(x)$  IS ITSELF. BUT THIS DOES NOT HELP US. WHAT IS  
THE DERIVATIVE OF  $g(y)$  THEN?

$$\textcircled{3} \quad g'(y) = \frac{1}{f'(x)}$$

$$\textcircled{4} \quad g'(y) = \frac{1}{f'(x)} = \frac{1}{f(x)} = \frac{1}{y}$$

Now, we can use the Fundamental  
Theorem of Calculus. It gives

$$\textcircled{5} \quad \int_1^\alpha \frac{1}{y} dy = g(\alpha) - g(1)$$

If we assume  $g(1) = 0$  here . . .

← Since we get this generally,\*

← we get this result, which  
shows that the derivative of  
the inverse function  $g(y)$  is  
explicitly given by  $\frac{1}{y}$ .

← Since we now know  $g'(y) = \frac{1}{y}$ ,  
function  $g(\alpha)$  is found to  
be a function obtained by  
integrating  $\frac{1}{y}$  from 1 to  $\alpha$ .

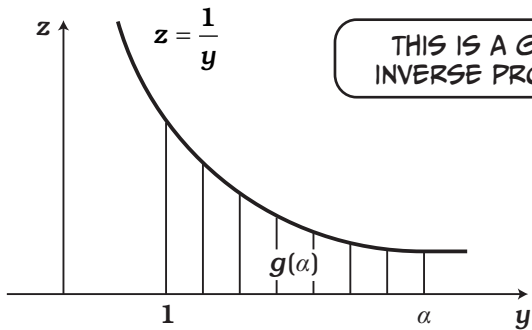


WE GET  $g(\alpha) = \int_1^\alpha \frac{1}{y} dy$

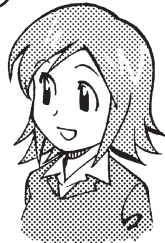
GOOD! NOW, LET'S DRAW THE GRAPH OF  $z = \frac{1}{y}$  !



\* As shown on page 75, if the inverse function of  $y = f(x)$  is  $x = g(y)$ ,  $f'(x) g'(y) = 1$ .

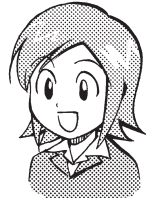


THIS IS A GRAPH OF INVERSE PROPORTION.



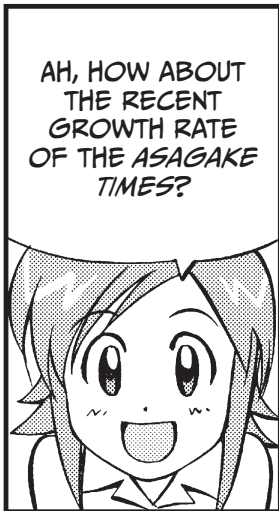
LET'S DEFINE  $g(\alpha)$  AS THE AREA BETWEEN THIS GRAPH AND THE Y-AXIS IN THE INTERVAL FROM 1 TO  $\alpha$ . THIS IS A WELL-DEFINED FUNCTION. IN OTHER WORDS,  $g(\alpha)$  IS STRICTLY DEFINED FOR ANY  $\alpha$ , WHETHER IT IS A FRACTION OR  $\sqrt{2}$ .

SINCE  $z = \frac{1}{y}$  IS AN EXPLICIT FUNCTION, THE AREA CAN BE ACCURATELY DETERMINED.

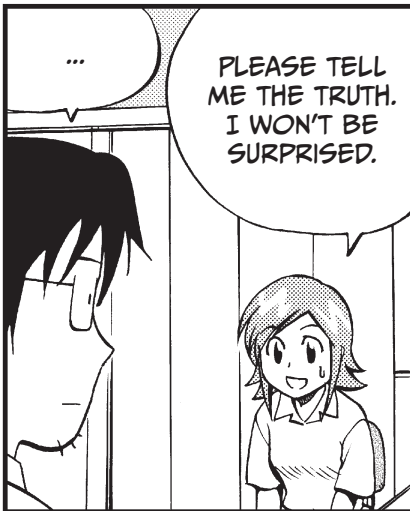


Since  $g(1) = \int_1^1 \frac{1}{y} dy = 0$ ,  $\int_1^\alpha \frac{1}{y} dy = g(\alpha) - g(1)$  which satisfies ⑤.

Thus, we have found out the inverse function  $g(y)$ , the area under the curve, which also gives the original function  $f(x)$ .



AH, HOW ABOUT THE RECENT GROWTH RATE OF THE ASAGAKE TIMES?



PLEASE TELL ME THE TRUTH. I WON'T BE SURPRISED.



YOU'RE CRYING! IS IT THAT BAD?

## SUMMARY OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

①  $\frac{f'(x)}{f(x)}$  is thought to be the growth rate.

②  $y = f(x)$  which satisfies  $\frac{f'(x)}{f(x)} = 1$  is the function that has a constant growth rate of 1.

This is an exponential function and satisfies

$$f'(x) = f(x)$$

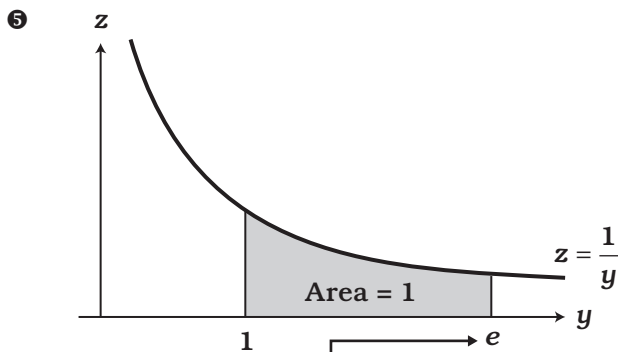
③ If the inverse function of  $y = f(x)$  is given by  $x = g(y)$ , we have

$$g'(y) = \frac{1}{y} \quad \star$$

④ If we define  $g(\alpha)$ , we can find the area of  $h(y) = \frac{1}{y}$ ,

$$g(\alpha) = \int_1^\alpha \frac{1}{y} dy$$

The inverse function of  $f(x)$  is the function that satisfies  $\star$  and  $g(1) = 0$ .



$e$  is an irrational number that is about 2.7178.

We define  $e$  (the base of the natural logarithm) as  $y$  that satisfies  $g(y) = 1$ . That is, it is the  $\alpha$  for which the area between the  $1/y$  curve and the  $y$ -axis in the interval from 1 to  $\alpha$  equals 1.

Since  $f(x)$  is an exponential function, we can write, using constant  $a_0$ ,

$$f(x) = a_0 a^x$$

Since  $f(g(1)) = f(0) = a_0 a^0 = a_0$  and  $f(g(1)) = 1$ , we get

$$f(g(1)) = 1 = a_0$$

And so we know

$$f(x) = a^x$$

Similarly, since

$$f(g(e)) = f(1) = a^1 \quad \text{and}$$

$$f(g(e)) = e$$

$$e = a^1$$

Thus, we have  $f(x) = e^x$ .

The inverse function  $g(y)$  of this is  $\log_e y$ , which can be simply written as  $\ln y$  ( $\ln$  stands for the natural logarithm).

Now let's rewrite ② through ④ in terms of  $e^x$  and  $\ln y$ .

$$\textcircled{6} \quad f'(x) = f(x) \Leftrightarrow (e^x)' = e^x$$

$$\textcircled{7} \quad g'(y) = \frac{1}{y} \Leftrightarrow (\ln y)' = \frac{1}{y}$$

$$\textcircled{8} \quad g(\alpha) = \int_1^\alpha \frac{1}{y} dy \Leftrightarrow \ln y = \int_1^y \frac{1}{y} dy$$

⑨ To define  $2^x$ , a function of bits, for any real number  $x$ , we look at

$$f(x) = e^{(\ln 2)x} \quad (x \text{ is any real number})$$

The reason is as follows. Because  $e^x$  and  $\ln y$  are inverse functions to each other,

$$e^{\ln 2} = 2$$

Therefore, for any natural number  $x$ , we have

$$f(x) = (e^{\ln 2})^x = 2^x$$

## MORE APPLICATIONS OF THE FUNDAMENTAL THEOREM

Other functions can be expressed in the form of  $f(x) = x^\alpha$ . Some of them are

$$\frac{1}{x} = x^{-1}, \frac{1}{x^2} = x^{-2}, \frac{1}{x^3} = x^{-3}, \dots$$

For such functions in general, the formula we found earlier holds true.

### FORMULA 4-2: THE POWER RULE OF DIFFERENTIATION

$$f(x) = x^\alpha \quad f'(x) = \alpha x^{\alpha-1}$$

EXAMPLE:

$$\text{For } f(x) = \frac{1}{x^3}, \quad f'(x) = (x^{-3})' = -3x^{-4} = -\frac{3}{x^4}$$

$$\text{For } f(x) = \sqrt[4]{x}, \quad f'(x) = \left(x^{\frac{1}{4}}\right)' = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$



PROOF:

Let's express  $f(x)$  in terms of  $e$ . Noting  $e^{\ln x} = x$ , we have

$$f(x) = x^\alpha = (e^{\ln x})^\alpha = e^{\alpha \ln x}$$

Thus,

$$\ln f(x) = \alpha \ln x$$

Differentiating both sides, remembering that the derivative of  $\ln w = \frac{1}{w}$ , and applying the chain rule,

$$\frac{1}{f(x)} \times f'(x) = \alpha \times \frac{1}{x}$$

Therefore,

$$f'(x) = \alpha \times \frac{1}{x} \times f(x) = \alpha \times \frac{1}{x} \times x^\alpha = \alpha x^{\alpha-1}$$

## INTEGRATION BY PARTS

If  $h(x) = f(x)g(x)$ , we get from the product rule of differentiation,

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Thus, since the function (the antiderivative) that gives  $f'(x)g(x) + f(x)g'(x)$  after differentiation is  $f(x)g(x)$ , we obtain from the Fundamental Theorem of Calculus,

$$\int_a^b \{f'(x)g(x) + f(x)g'(x)\} dx = f(b)g(b) - f(a)g(a)$$

Using the sum rule of integration, we obtain the following formula.

### FORMULA 4-3: INTEGRATION BY PARTS

$$\int_a^b f'(x)g(x) dx + \int_a^b f(x)g'(x) dx = f(b)g(b) - f(a)g(a)$$

As an example, let's calculate:

$$\int_0^\pi x \sin x dx$$

We guess the integral's answer will be a similar form to  $x \cos x$ , so we say  $f(x) = x$  and  $g(x) = \cos x$ . So we try,

$$\int_0^\pi x' \cos x dx + \int_0^\pi x (\cos x)' dx = f(x)g(x) \Big|_0^\pi$$

We can evaluate that

$$= f(\pi)g(\pi) - f(0)g(0)$$

Substituting in our original functions of  $f(x)$  and  $g(x)$ , we find that

$$= \pi \cos \pi - 0 \cos 0 = \pi(-1) - 0 = -\pi$$

We can use this result in our first equation.

$$\int_0^\pi x' \cos x dx + \int_0^\pi x (\cos x)' dx = -\pi$$

We then get:

$$\int_0^{\pi} \cos x \, dx + \int_0^{\pi} x(-\sin x) \, dx = -\pi$$

Rearranging it further by pulling out the negatives, we find:

$$\int_0^{\pi} \cos x \, dx - \int_0^{\pi} x \sin x \, dx = -\pi$$

And you can see here that we have the original integral, but now we have it in terms that we can actually solve! We solve for our original function:

$$\int_0^{\pi} x \sin x \, dx = \int_0^{\pi} \cos x \, dx + \pi$$

Remember that  $\int \cos x \, dx = \sin x$ , and you can see that

$$\int_0^{\pi} x \sin x \, dx = \sin x \Big|_0^{\pi} + \pi$$

$$= \sin \pi - \sin 0 + \pi$$

$$= 0 - 0 + \pi = \pi$$

There you have it.



## EXERCISES

1.  $\tan x$  is a function defined as  $\sin x / \cos x$ . Obtain the derivative of  $\tan x$ .
2. Calculate

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} \, dx$$

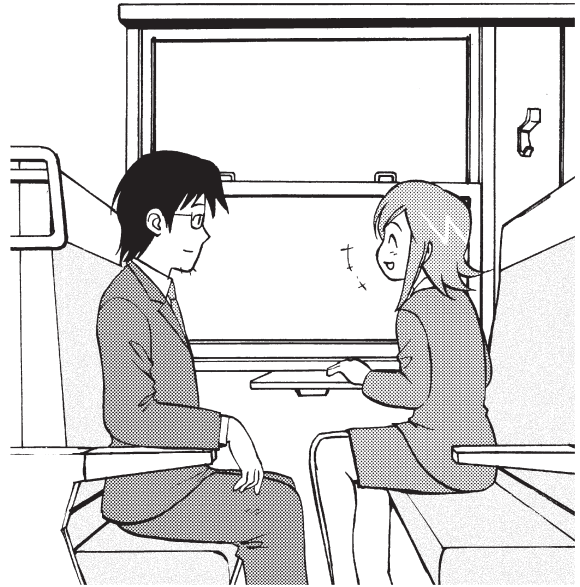
3. Obtain such  $x$  that makes  $f(x) = xe^x$  minimum.
4. Calculate

$$\int_1^e 2x \ln x \, dx$$

A clue: Suppose  $f(x) = x^2$  and  $g(x) = \ln x$ , and use integration by parts.

5

LET'S LEARN ABOUT  
TAYLOR EXPANSIONS!



THE ASAGAKE TIMES  
MAIN OFFICE

あさがけ新聞社

WOW! WHAT  
AN OFFICE.

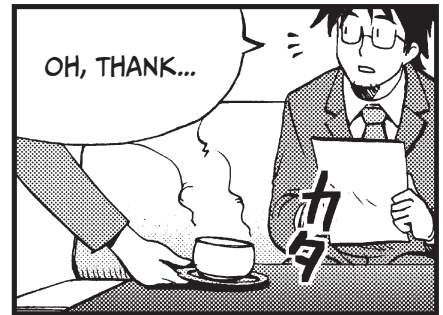
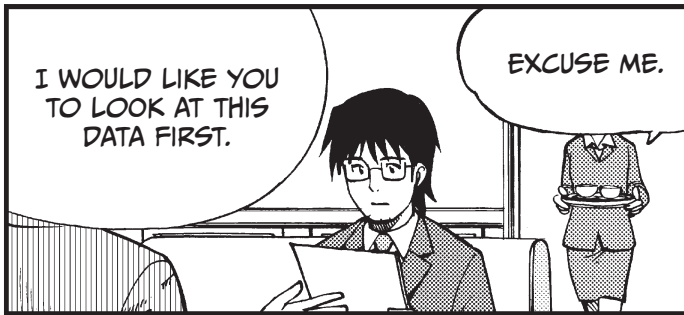
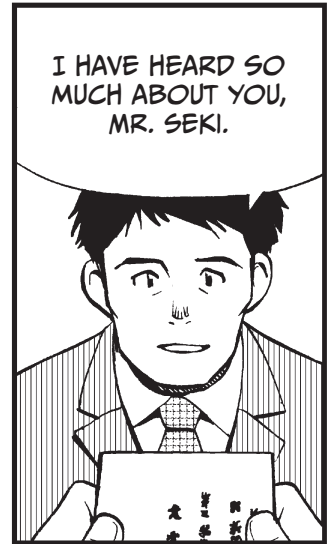
I WANNA  
WORK HERE!

I HAVE A MEETING.  
WILL YOU WAIT FOR  
ME IN THE LOBBY?

WHAT...WILL I BE  
A NUISANCE?

SMIRK

# IMITATING WITH POLYNOMIALS





NORIKO, WHAT ARE YOU DOING? YOU LOOK SUSPICIOUS.

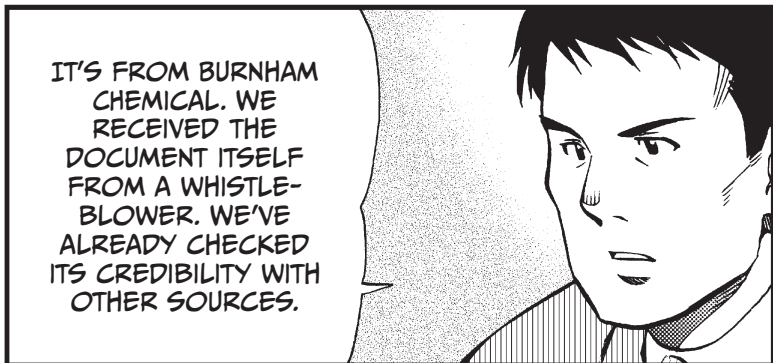
SO DUSTY!  
HMM...

WIPE WIPE

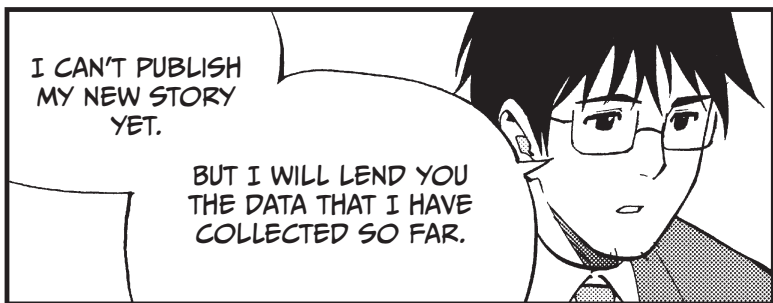


THIS IS THE SAME DATA THAT YOU USED IN YOUR ARTICLE, ISN'T IT?

AH, YES...WHAT'S THE SOURCE OF THIS DATA?

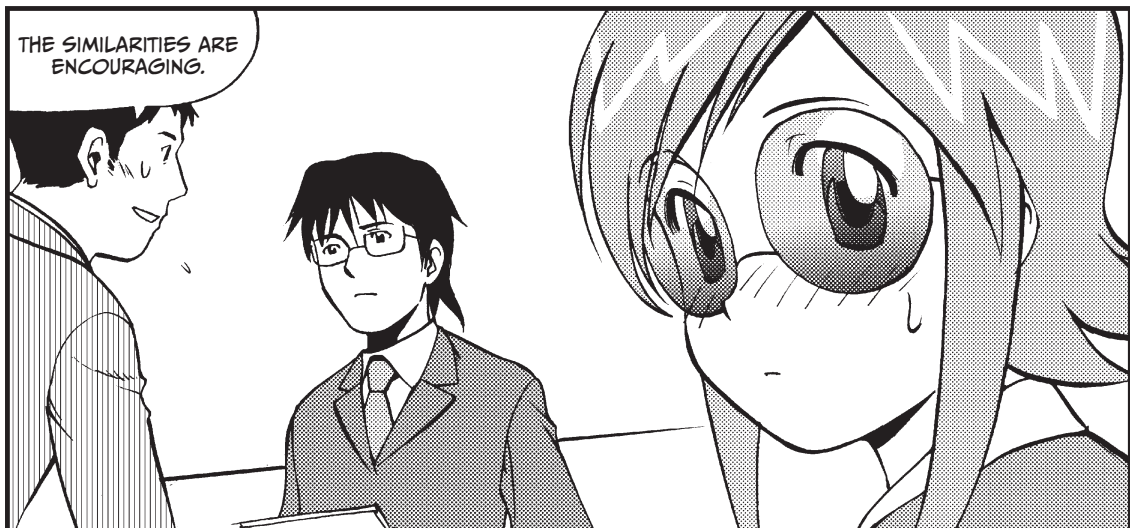


IT'S FROM BURNHAM CHEMICAL. WE RECEIVED THE DOCUMENT ITSELF FROM A WHISTLE-BLOWER. WE'VE ALREADY CHECKED ITS CREDIBILITY WITH OTHER SOURCES.

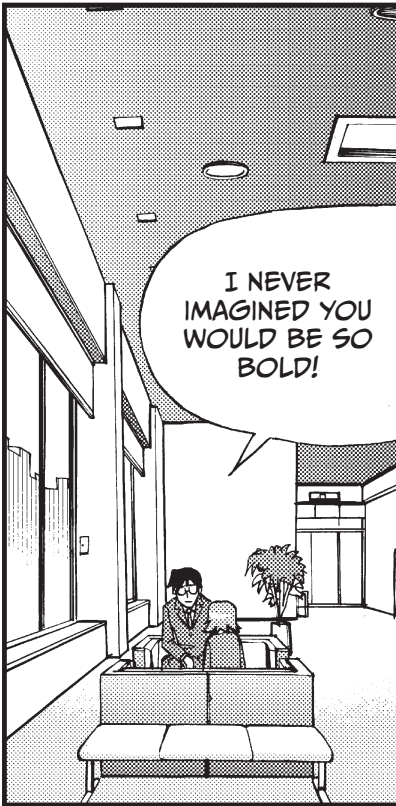


I CAN'T PUBLISH MY NEW STORY YET.

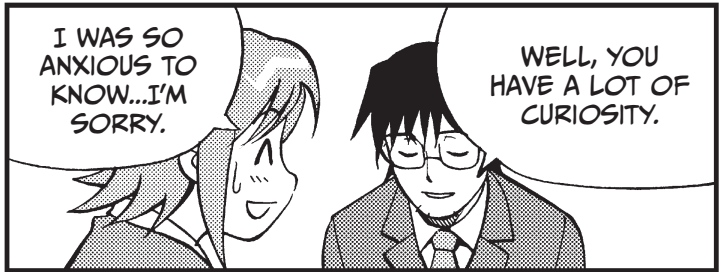
BUT I WILL LEND YOU THE DATA THAT I HAVE COLLECTED SO FAR.



THE SIMILARITIES ARE ENCOURAGING.

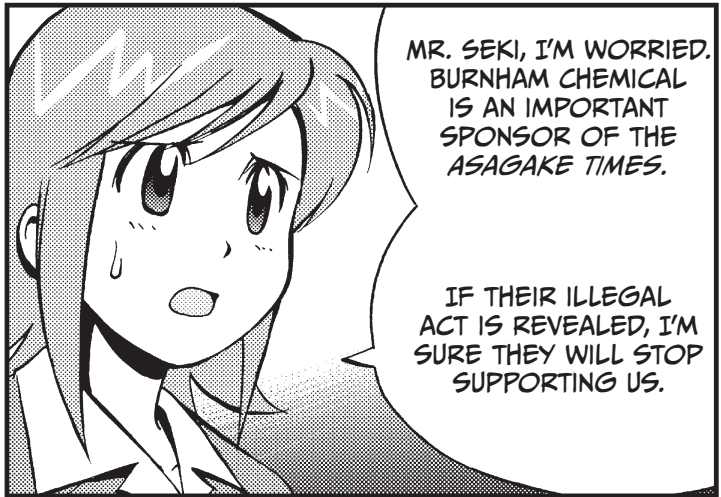


I NEVER  
IMAGINED YOU  
WOULD BE SO  
BOLD!



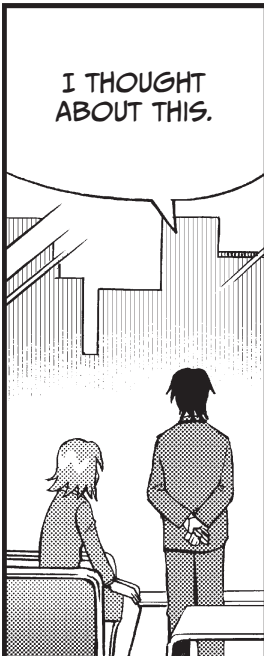
I WAS SO  
ANXIOUS TO  
KNOW...I'M  
SORRY.

WELL, YOU  
HAVE A LOT OF  
CURIOSITY.

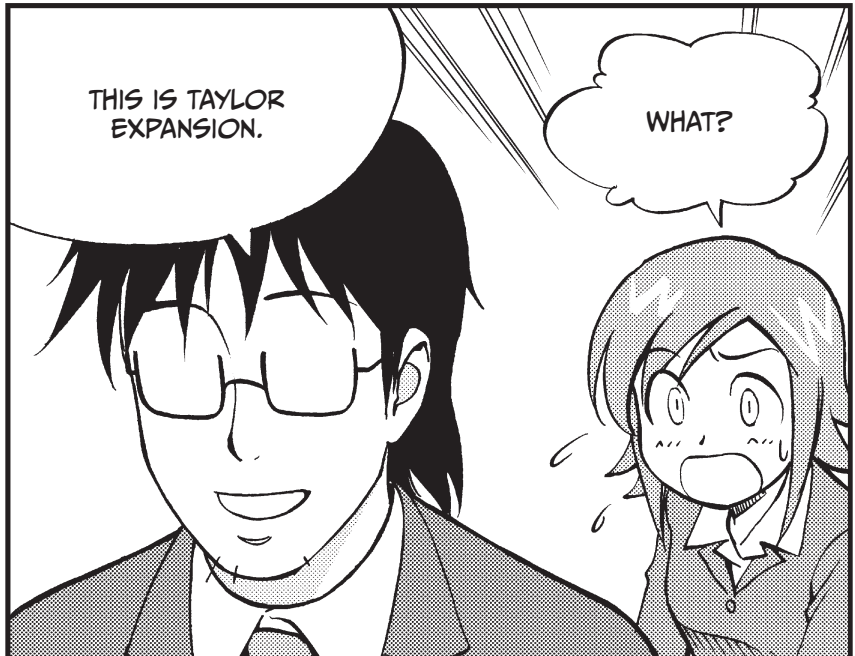


MR. SEKI, I'M WORRIED.  
BURNHAM CHEMICAL  
IS AN IMPORTANT  
SPONSOR OF THE  
ASAGAKE TIMES.

IF THEIR ILLEGAL  
ACT IS REVEALED, I'M  
SURE THEY WILL STOP  
SUPPORTING US.

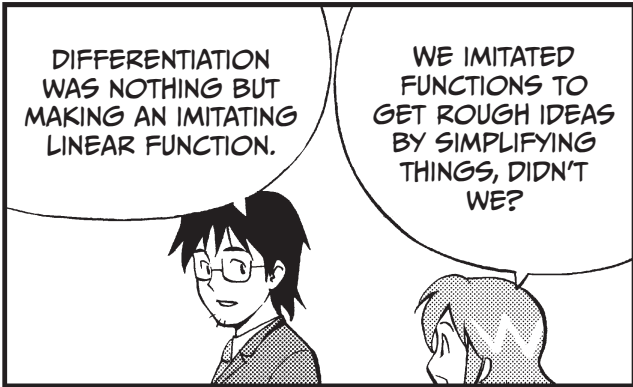


I THOUGHT  
ABOUT THIS.



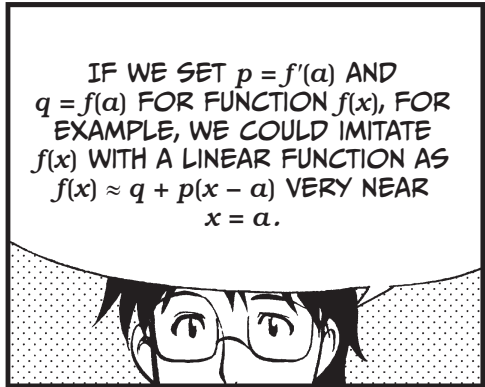
THIS IS TAYLOR  
EXPANSION.

WHAT?

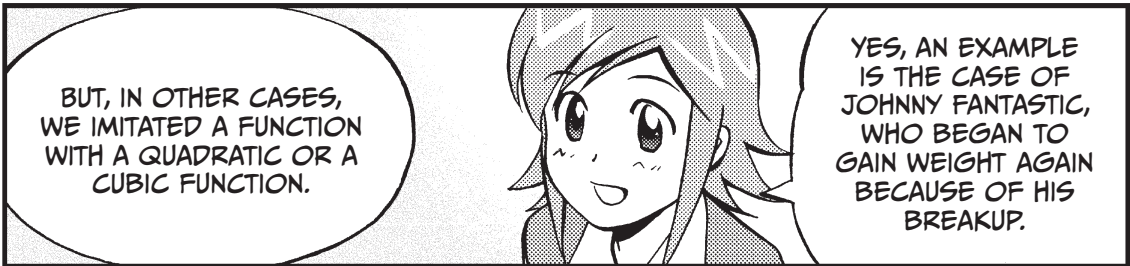


DIFFERENTIATION WAS NOTHING BUT MAKING AN IMITATING LINEAR FUNCTION.

WE IMITATED FUNCTIONS TO GET ROUGH IDEAS BY SIMPLIFYING THINGS, DIDN'T WE?

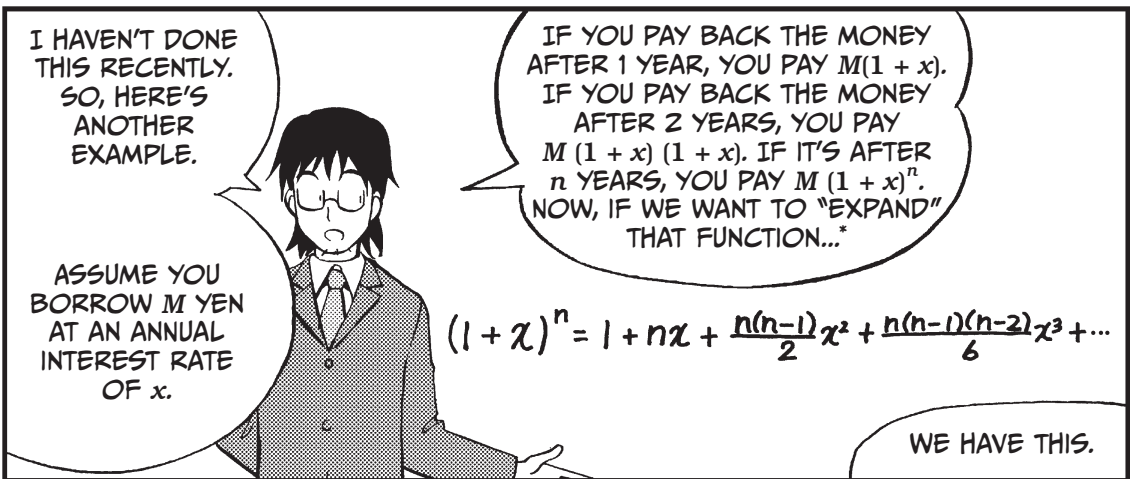


IF WE SET  $p = f'(a)$  AND  $q = f(a)$  FOR FUNCTION  $f(x)$ , FOR EXAMPLE, WE COULD IMITATE  $f(x)$  WITH A LINEAR FUNCTION AS  $f(x) \approx q + p(x - a)$  VERY NEAR  $x = a$ .



BUT, IN OTHER CASES, WE IMITATED A FUNCTION WITH A QUADRATIC OR A CUBIC FUNCTION.

YES, AN EXAMPLE IS THE CASE OF JOHNNY FANTASTIC, WHO BEGAN TO GAIN WEIGHT AGAIN BECAUSE OF HIS BREAKUP.



I HAVEN'T DONE THIS RECENTLY. SO, HERE'S ANOTHER EXAMPLE.

ASSUME YOU BORROW  $M$  YEN AT AN ANNUAL INTEREST RATE OF  $x$ .

IF YOU PAY BACK THE MONEY AFTER 1 YEAR, YOU PAY  $M(1 + x)$ . IF YOU PAY BACK THE MONEY AFTER 2 YEARS, YOU PAY  $M(1 + x)(1 + x)$ . IF IT'S AFTER  $n$  YEARS, YOU PAY  $M(1 + x)^n$ . NOW, IF WE WANT TO "EXPAND" THAT FUNCTION...\*

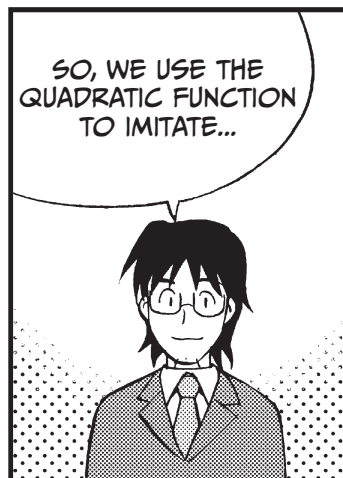
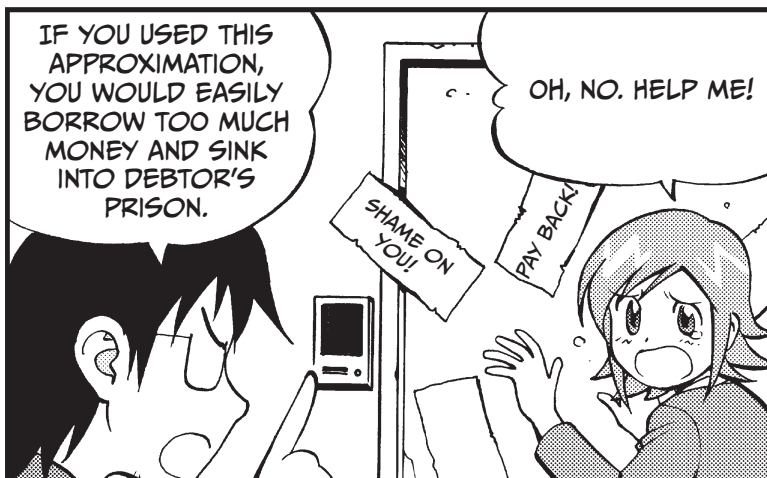
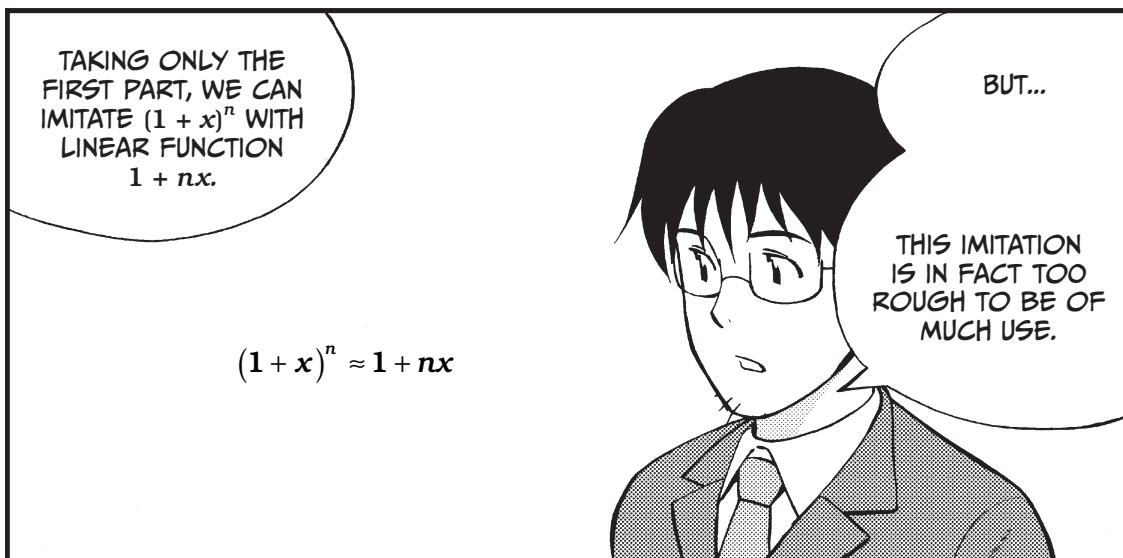
$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \dots$$

WE HAVE THIS.

$$(1 + x)^n = 1 + {}_n C_1 x + {}_n C_2 x^2 + {}_n C_3 x^3 + \dots + {}_n C_n x^n$$

\* THIS IS THE FORMULA OF BINOMIAL EXPANSION, WHERE  ${}_n C_r = \frac{n!}{r!(n-r)!}$  AND  ${}_n C_1 = n$

$${}_n C_2 = \frac{n(n-1)}{2}, {}_n C_3 = \frac{n(n-1)(n-2)}{6}, \dots, {}_n C_r = \frac{n(n-1)\dots\{n-(r-1)\}}{r!}$$



FORMULA 5-1: THE FORMULA OF QUADRATIC APPROXIMATION

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2$$

IF WE MODIFY THIS EXPRESSION  
A LITTLE, WE GET A VERY  
INTERESTING LAW.



For any pair of  $n$  and  $x$  that satisfy  $nx = 0.7$ , we get

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2 \approx 1 + nx + \frac{1}{2}(nx)^2 - \frac{1}{2}nx^2$$

$\approx 1 + 0.7 + \frac{1}{2} \times 0.7^2 = 1.945 \approx 2$

Nearly zero, so we neglect it.

In short, if  $nx = 0.7$ ,  $(1+x)^n$  is almost 2. This can be written as a law as follows.

LAW OF DEBT HELL

When years to repay loan  $\times$  interest rate = 0.7, the amount you will repay is about twice as much as you borrowed.

- ABOUT TWICE IF BORROWED FOR  
35 YEARS AT 2 PERCENT
- ABOUT TWICE IF BORROWED FOR  
7 YEARS AT 10 PERCENT
- ABOUT TWICE IF BORROWED FOR  
2 YEARS AT 35 PERCENT



OH, NO!!  
THIS IS TERRIBLE!!

THE TERMS  $x^n$  FOR WHICH  $n$  IS MORE THAN 1 ARE CALLED HIGH-DEGREE TERMS.



IMITATING A FUNCTION WITH A QUADRATIC (2ND-DEGREE) FUNCTION IN THIS WAY OFTEN ALLOWS US TO FIND INTERESTING THINGS. NOW, LET'S CONSIDER IMITATING A FUNCTION WITH A HIGHER-DEGREE POLYNOMIAL. IN FACT, IT IS KNOWN THAT WE CAN MAKE THE EXACT FUNCTION, INSTEAD OF AN IMITATION, WITH AN INFINITE-DEGREE POLYNOMIAL.

For example, if we set  $f(x) = \frac{1}{1-x}$ , we get

$$\textcircled{1} f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \quad (\text{continues infinitely})$$

Note this is = instead of  $\approx$ .



THIS IS A MISTAKE, ISN'T IT? IT CAN'T BE EQUAL TO!

I THOUGHT YOU WOULD SAY THAT. LET'S CALCULATE IT.



Suppose  $x = 0.1$ . We get

$$f(0.1) = \frac{1}{1-0.1} = \frac{1}{0.9} = \frac{10}{9}$$

$$\begin{aligned} \text{Right side} &= 1 + 0.1 + 0.1^2 + 0.1^3 + 0.1^4 + \dots \\ &= 1 + 0.1 + 0.01 + 0.001 + 0.0001 + \dots \\ &= 1.111111\dots \end{aligned}$$

If we actually calculate  $10/9$  by long division, we will obtain the same result.

$$\begin{array}{r} 1.111\dots \\ 9 \overline{) 10} \\ \underline{9} \phantom{00} \\ 10 \phantom{00} \\ \underline{9} \phantom{00} \\ 10 \phantom{00} \\ \underline{9} \phantom{00} \\ 10 \phantom{00} \\ \underline{9} \phantom{00} \end{array}$$

When a general function  $f(x)$  (provided it is differentiable infinitely many times) can be expressed as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$$

the right side is called the *Taylor expansion* of  $f(x)$  (about  $x = 0$ ).



THIS MEANS THAT  $f(x)$  PERFECTLY COINCIDES WITH AN INFINITE-DEGREE POLYNOMIAL IN A DEFINITE INTERVAL INCLUDING  $x = 0$ . IT SHOULD BE NOTED, HOWEVER, THAT THE RIGHT SIDE MAY BECOME MEANINGLESS BECAUSE IT MAY NOT HAVE A SINGLE DEFINED VALUE OUTSIDE THE INTERVAL.

FOR EXAMPLE, SUBSTITUTING  $x = 2$  IN BOTH SIDES OF EXPRESSION ❶,

$$\text{Left side} = \frac{1}{1-2} = -1$$

$$\text{Right side} = 1 + 2 + 4 + 8 + 16 + \dots$$

It turns out that expression ❶ is correct for all  $x$  satisfying  $-1 < x < 1$ , which is the allowed interval of a Taylor expansion. In technical terms, the interval  $-1 < x < 1$  is called the *circle of convergence*.

SEE? THE TWO SIDES ARE NOT EQUAL.



## HOW TO OBTAIN A TAYLOR EXPANSION

When we have

$$\textcircled{2} \quad f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$$

let's find the coefficient  $a_n$ .

Substituting  $x = 0$  in the above equation and noting  $f(0) = a_0$ , we find that the 0th-degree coefficient  $a_0$  is  $f(0)$ .

We then differentiate  $\textcircled{2}$ .

$$\textcircled{3} \quad f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$

Substituting  $x = 0$  in  $\textcircled{3}$  and noting  $f'(0) = a_1$ , we find that the 1st-degree coefficient  $a_1$  is  $f'(0)$ .

We differentiate  $\textcircled{3}$  to get

$$\textcircled{4} \quad f''(x) = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots$$

Substituting  $x = 0$  in  $\textcircled{4}$ , we find that the 2nd-degree coefficient  $a_2$  is  $\frac{1}{2}f''(0)$ .

Differentiating  $\textcircled{4}$ , we get

$$f'''(x) = 6a_3 + \dots + n(n-1)(n-2)a_nx^{n-3} + \dots$$

From this, we find that the 3rd-degree coefficient  $a_3$  is  $\frac{1}{6}f'''(0)$ .

Repeating this differentiation operation  $n$  times, we get

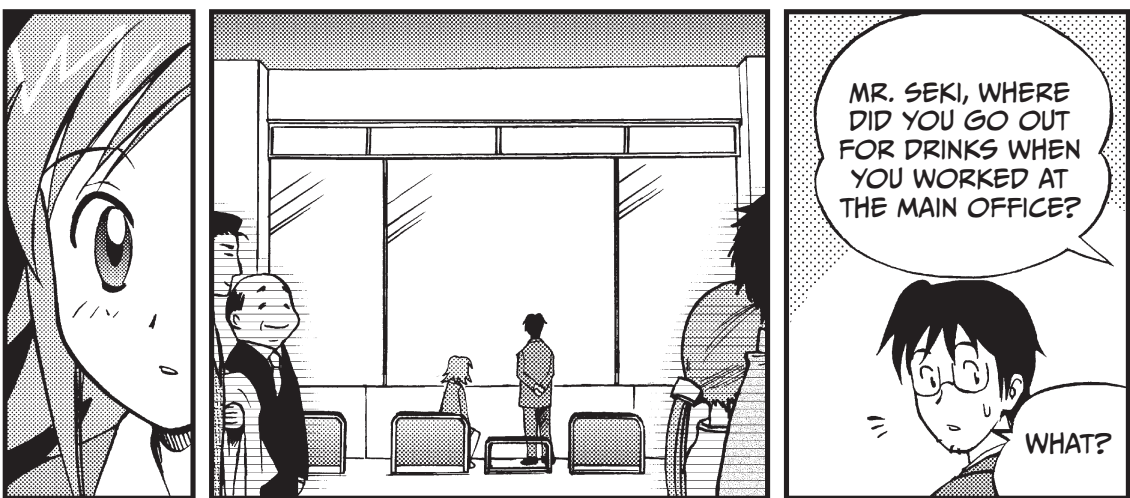
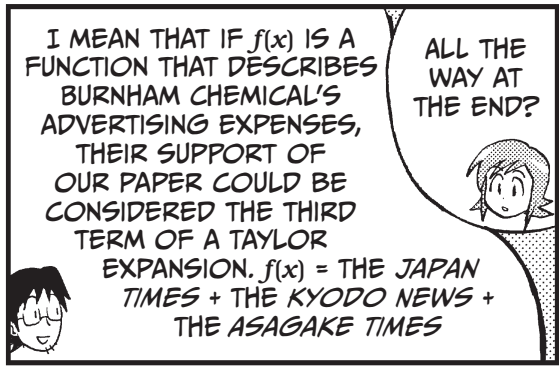
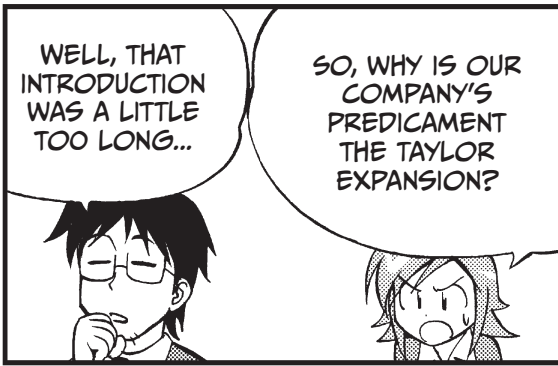
$$f^{(n)}(x) = n(n-1)\dots \times 2 \times 1 a_n + \dots$$

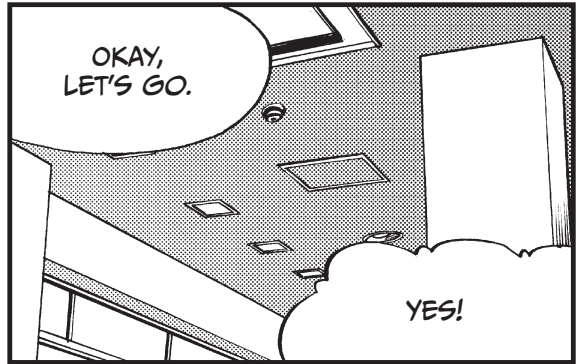
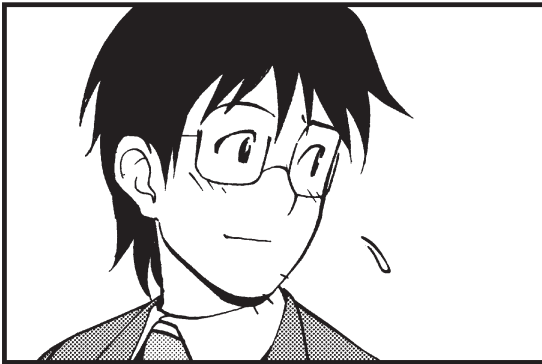
where  $f^{(n)}(x)$  is the expression obtained after differentiating  $f(x)$   $n$  times.

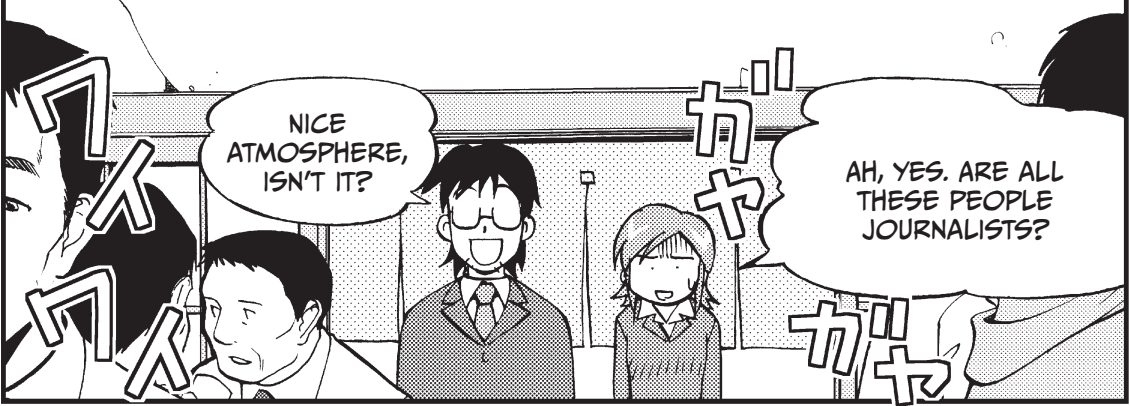
From this result, we find

$$\textit{nth-degree coefficient} \quad a_n = \frac{1}{n!} f^{(n)}(0)$$

$n!$  is read “ $n$  factorial” and means  $n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$ .

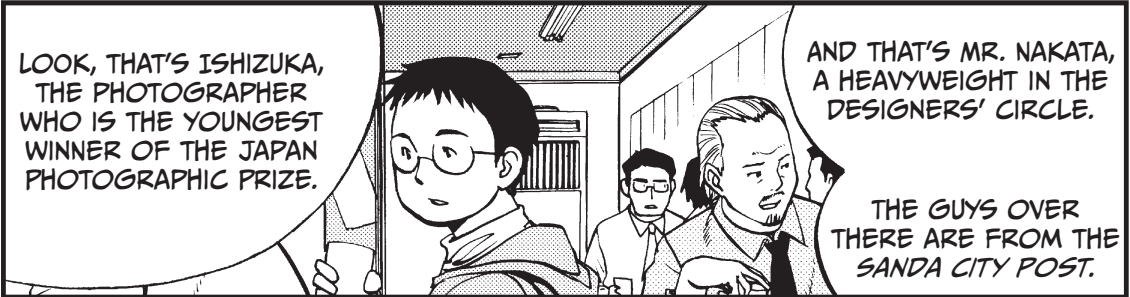






NICE  
ATMOSPHERE,  
ISN'T IT?

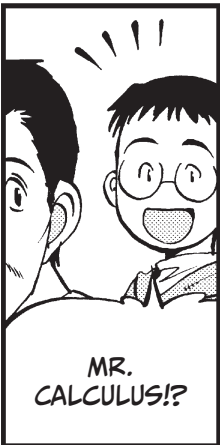
カヤ  
カヤ  
AH, YES. ARE ALL  
THESE PEOPLE  
JOURNALISTS?



LOOK, THAT'S ISHIZUKA,  
THE PHOTOGRAPHER  
WHO IS THE YOUNGEST  
WINNER OF THE JAPAN  
PHOTOGRAPHIC PRIZE.

AND THAT'S MR. NAKATA,  
A HEAVYWEIGHT IN THE  
DESIGNERS' CIRCLE.

THE GUYS OVER  
THERE ARE FROM THE  
SANDA CITY POST.

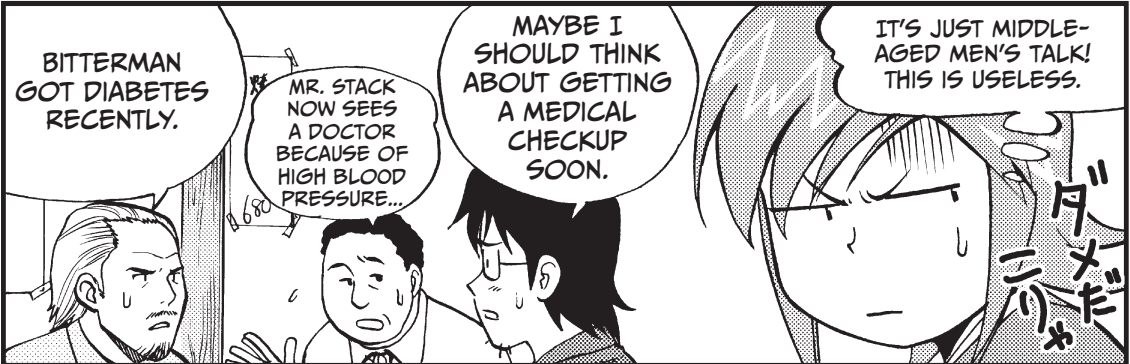


HEY, CALCULUS,  
LONG TIME NO  
SEE. JOIN US.

HELLO.

WHAT A  
NICKNAME! BUT,  
IT CERTAINLY  
FITS.

LET'S SEE. I  
HOPE I CAN  
LISTEN TO THEIR  
PROFESSIONAL  
DISCUSSION.



BITTERMAN  
GOT DIABETES  
RECENTLY.

MR. STACK  
NOW SEES  
A DOCTOR  
BECAUSE OF  
HIGH BLOOD  
PRESSURE...

MAYBE I  
SHOULD THINK  
ABOUT GETTING  
A MEDICAL  
CHECKUP  
SOON.

IT'S JUST MIDDLE-  
AGED MEN'S TALK!  
THIS IS USELESS.

ダニ  
ダニ

### FORMULA 5-2: THE FORMULA OF TAYLOR EXPANSION

If  $f(x)$  has a Taylor expansion about  $x = 0$ , it is given by

$$f(x) = f(0) + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \dots + \frac{1}{n!} f^{(n)}(0)x^n + \dots$$

For the above,

$f(0)$	0th-degree constant term	$a_0 = f(0)$
$f'(0)x$	1st-degree term	$a_1 = f'(0)$
$\frac{1}{2!} f''(0)x^2$	2nd-degree term	$a_2 = \frac{1}{2} f''(0)$
$\frac{1}{3!} f'''(0)x^3$	3rd-degree term	$a_3 = \frac{1}{6} f'''(0)$

For the moment, we forget about the conditions for having Taylor expansion and the circle of convergence.

Using this formula, we check **1** on page 153.

$$f(x) = \frac{1}{1-x}, f'(x) = \frac{1}{(1-x)^2}, f''(x) = \frac{2}{(1-x)^3}, f'''(x) = \frac{6}{(1-x)^4}, \dots$$

$$f(0) = 1, f'(0) = 1, f''(0) = 2, f'''(0) = 6, \dots, f^{(n)}(0) = n!$$

Thus, we have

$$\begin{aligned} f(x) &= f(0) + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \dots + \frac{1}{n!} f^{(n)}(0)x^n + \dots \\ &= 1 + x + \frac{1}{2!} \times 2x^2 + \frac{1}{3!} \times 6x^3 + \dots + \frac{1}{n!} n!x^n + \dots \\ &= 1 + x + x^2 + x^3 + \dots + x^n + \dots \end{aligned}$$

THEY  
COINCIDE!



THE FORMULA ABOVE IS FOR AN INFINITE-DEGREE POLYNOMIAL THAT COINCIDES WITH THE ORIGINAL NEAR  $x = 0$ . THE FORMULA FOR A POLYNOMIAL THAT COINCIDES NEAR  $x = a$  IS GENERALLY GIVEN AS FOLLOWS. TRY THE EXERCISE ON PAGE 178 TO CHECK THIS!

$$\begin{aligned} f(x) &= f(a) + \frac{1}{1!} f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 \\ &\quad + \frac{1}{3!} f'''(a)(x-a)^3 + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \dots \end{aligned}$$

TAYLOR EXPANSION IS A SUPERIOR IMITATING FUNCTION.



## TAYLOR EXPANSION OF VARIOUS FUNCTIONS

### [1] TAYLOR EXPANSION OF A SQUARE ROOT

We set  $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$ .

Thus, from  $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$$f''(x) = -\frac{1}{2} \times \frac{1}{2} (1+x)^{-\frac{3}{2}}$$

$$f'''(x) = \frac{1}{2} \times \frac{1}{2} \times \frac{3}{2} (1+x)^{-\frac{5}{2}}, \dots$$

$$f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}, f'''(0) = \frac{3}{8}, \dots$$

$$\begin{aligned} f(x) &= \sqrt{1+x} \\ &= 1 + \frac{1}{2}x + \frac{1}{2!} \times \left(-\frac{1}{4}\right)x^2 + \frac{1}{3!} \times \frac{3}{8}x^3 + \dots \end{aligned}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \dots$$

### [3] TAYLOR EXPANSION OF LOGARITHMIC FUNCTION $\ln(1+x)$

We set  $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = -(1+x)^{-2}, f^{(3)}(x) = 2(1+x)^{-3},$$

$$f^{(4)}(x) = -6(1+x)^{-4}, \dots$$

$$f(0) = 0, f'(0) = 1, f''(0) = -1, f^{(3)}(0) = 2!,$$

$$f^{(4)}(0) = -3!, \dots$$

Thus, we have

$$\ln(1+x) =$$

$$0 + x - \frac{1}{2}x^2 + \frac{1}{3!} \times 2!x^3 - \frac{1}{4}3!x^4 + \dots$$

$$\ln(1+x) =$$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n+1} \frac{1}{n}x^n + \dots$$

### [2] TAYLOR EXPANSION OF EXPONENTIAL FUNCTION $e^x$

If we set  $f(x) = e^x$ ,

$$f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x, \dots$$

So, from

$$\begin{aligned} e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots \\ &\quad + \frac{1}{n!}x^n + \dots \end{aligned}$$

Substituting  $x = 1$ , we get

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots$$

IN CHAPTER 4, WE LEARNED THAT  $e$  IS ABOUT 2.7. HERE, WE HAVE OBTAINED THE EXPRESSION TO CALCULATE IT EXACTLY.



### [4] TAYLOR EXPANSION OF TRIGONOMETRIC FUNCTIONS

We set  $f(x) = \cos x$ .

$$\begin{aligned} f'(x) &= -\sin x, f''(x) = -\cos x, f^{(3)}(x) \\ &= \sin x, f^{(4)}(x) = \cos x, \dots \end{aligned}$$

From

$$f(0) = 1, f'(0) = 0, f''(0) = -1,$$

$$f^{(3)}(0) = 0, f^{(4)}(0) = 1, \dots$$

Thus,

$$\cos x = 1 + 0x - \frac{1}{2!} \times 1 \times x^2 + \frac{1}{3!} \times 0 \times x^3 + \frac{1}{4!} \times 1 \times x^4 + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots$$

Similarly,

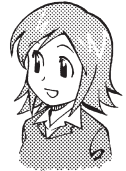
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots + (-1)^{n-1} \frac{1}{(2n-1)!}x^{2n-1} + \dots$$

# WHAT DOES TAYLOR EXPANSION TELL US?



TAYLOR EXPANSION REPLACES COMPLICATED FUNCTIONS WITH POLYNOMIALS. CAN YOU DRAW THE GRAPH OF, FOR EXAMPLE,  $\ln(1+x)$ ?

AFTER ALL, IT IS NECESSARY TO APPROXIMATE OR IMITATE FUNCTIONS TO UNCOVER THEIR COMPLICATED WORLD, ISN'T IT?



LET'S USE  $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ , AN EXAMPLE GIVEN ABOVE, TO SEE WHAT WE CAN GAIN FROM A TAYLOR EXPANSION.



$$\ln(1+x) = \underbrace{0}_{\text{0th degree approx.}} + \underbrace{x}_{\text{1 Linear approx.}} - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

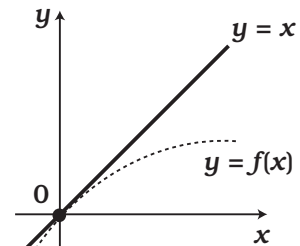
3 Cubic approx.  
2 Quadratic approx.

FIRST, 0th-DEGREE APPROXIMATION.  $\ln(1+x) \approx 0$  NEAR  $x=0$ . WHAT DOES THIS MEAN?

AH, WELL...IT MEANS THAT THE VALUE OF  $f(x)$  IS 0 AT  $x=0$  AND IT PASSES THROUGH POINT  $(0, 0)$ .



THAT'S RIGHT. NEXT IS LINEAR APPROXIMATION. YOU SEE THAT  $y=f(x)$  ROUGHLY RESEMBLES  $y=x$  NEAR  $x=0$ ? SO, THIS MEANS THAT THE FUNCTION IS INCREASING AT  $x=0$ . (NOTE: THE EQUATION OF A TANGENT LINE = LINEAR APPROXIMATION.)

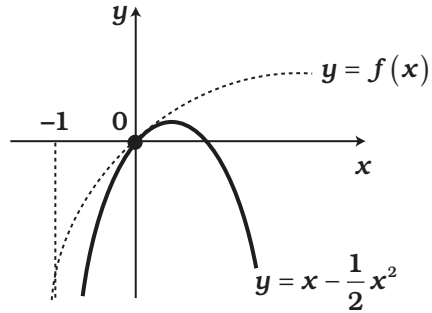




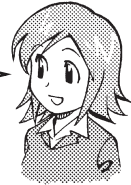
WE'LL NOW TAKE ONE MORE STEP TO QUADRATIC APPROXIMATION. LET'S CONSIDER THE GRAPH OF

$$\ln(1+x) \approx x - \frac{1}{2}x^2$$

AROUND  $x = 0$ . NORIKO, WHAT DOES THIS MEAN?



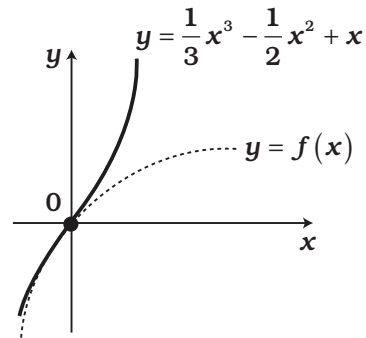
THIS MEANS THAT  $y = f(x)$  ROUGHLY RESEMBLES  $y = x - \frac{1}{2}x^2$  NEAR  $x = 0$  AND ITS GRAPH IS CONCAVE DOWN AT  $x = 0$ . (QUADRATIC APPROXIMATION ALLOWS US TO FIND HOW IT IS CURVED AT  $x = a$ .)



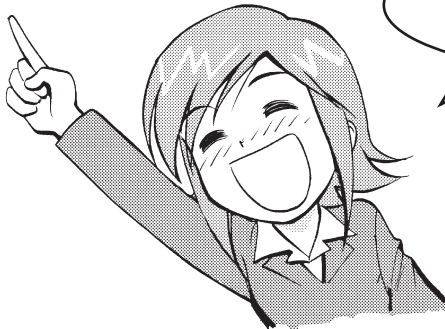
LET'S USE CUBIC APPROXIMATION AS THE LAST PUSH!! NEAR  $x = 0$ ,

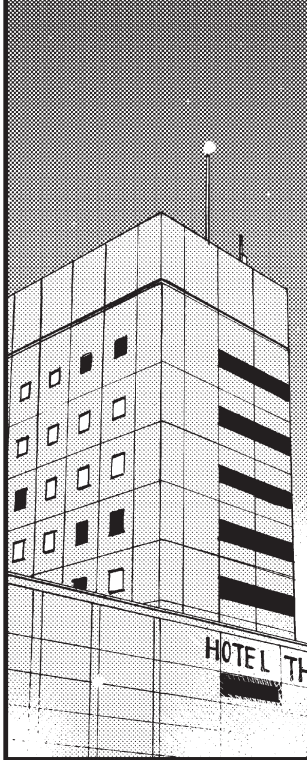
$$\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

(CUBIC APPROXIMATION FURTHER CORRECTS THE ERROR IN QUADRATIC APPROXIMATION.)



NOW, MR. SEKI, ON TO THE NEXT BAR!



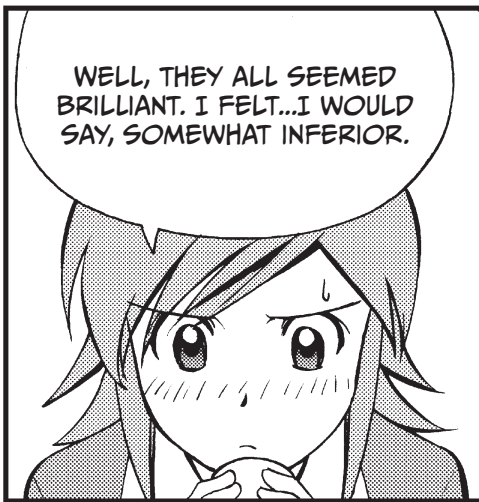


THIS IS BETTER! WE CAN TALK MORE QUIETLY AT THIS HOTEL BAR.

WE SHOULD HAVE JUST COME HERE IN THE FIRST PLACE.



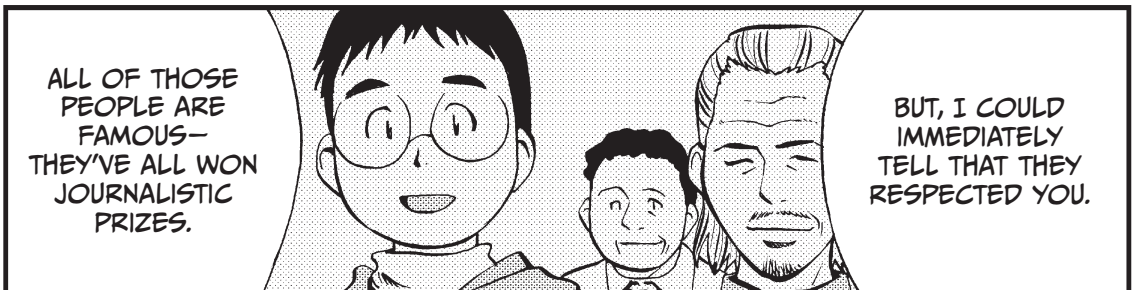
YOU COULD HAVE TALKED MORE WITH THE GUYS AT THAT PUB.



WELL, THEY ALL SEEMED BRILLIANT. I FELT...I WOULD SAY, SOMEWHAT INFERIOR.

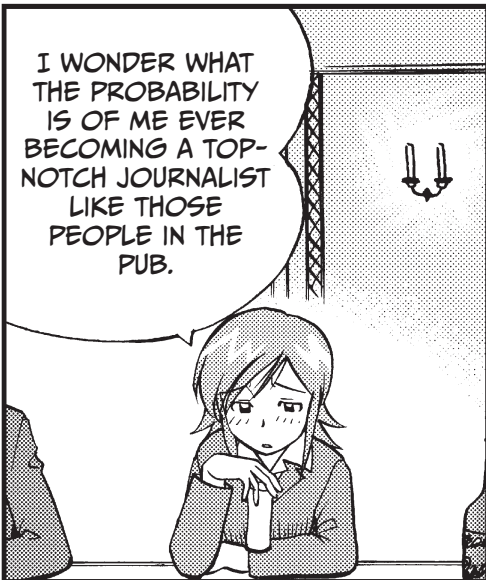
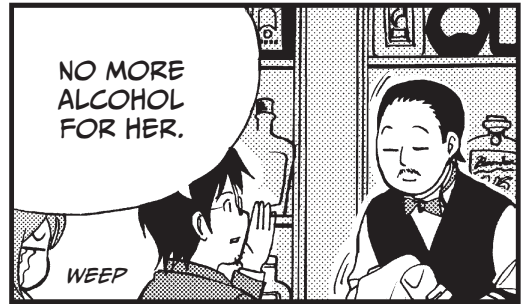


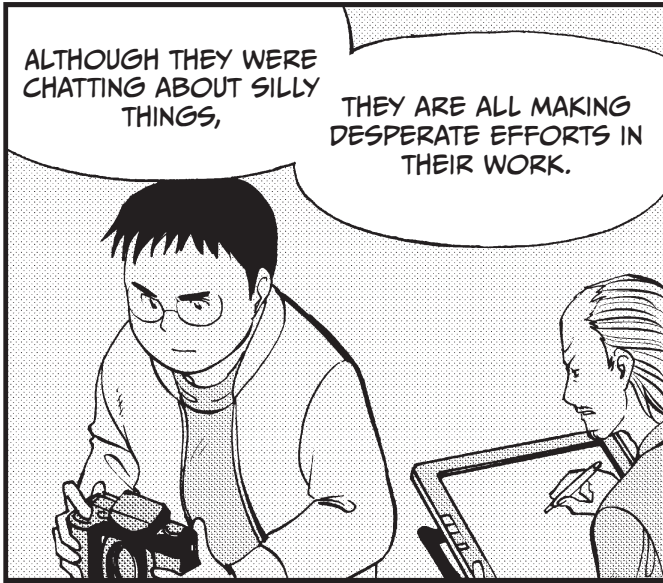
SO WHAT'S YOUR DEAL, MR. SEKI?



ALL OF THOSE PEOPLE ARE FAMOUS— THEY'VE ALL WON JOURNALISTIC PRIZES.

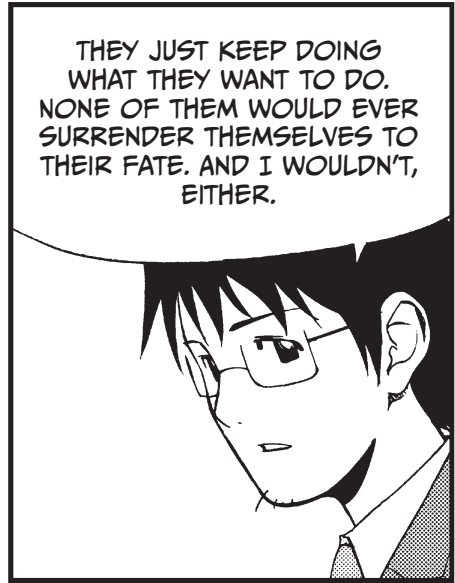
BUT, I COULD IMMEDIATELY TELL THAT THEY RESPECTED YOU.



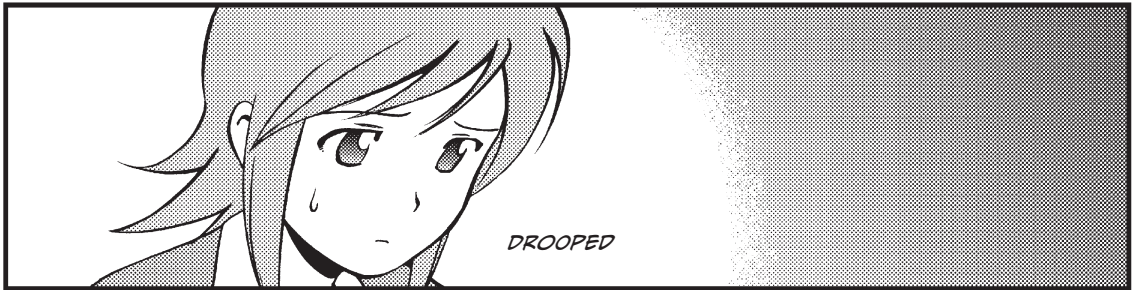


ALTHOUGH THEY WERE CHATTING ABOUT SILLY THINGS,

THEY ARE ALL MAKING DESPERATE EFFORTS IN THEIR WORK.



THEY JUST KEEP DOING WHAT THEY WANT TO DO. NONE OF THEM WOULD EVER SURRENDER THEMSELVES TO THEIR FATE. AND I WOULDN'T, EITHER.

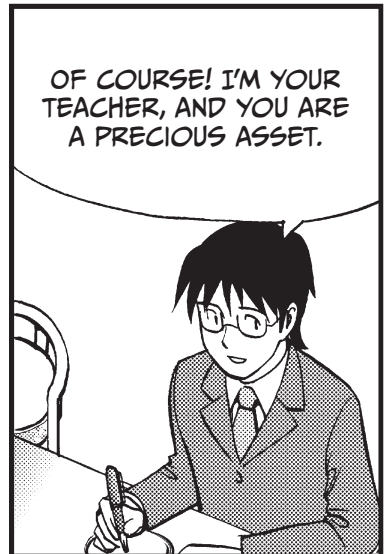


DROOPED

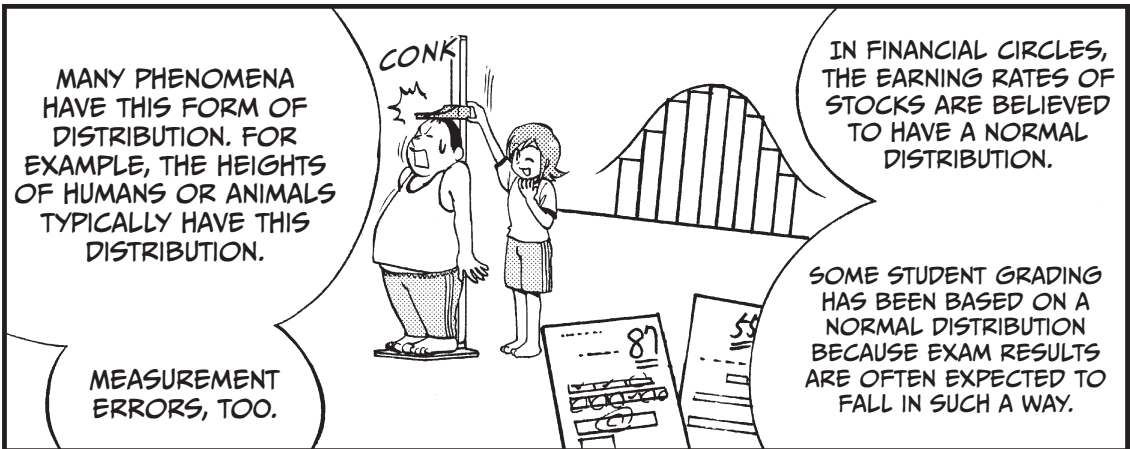
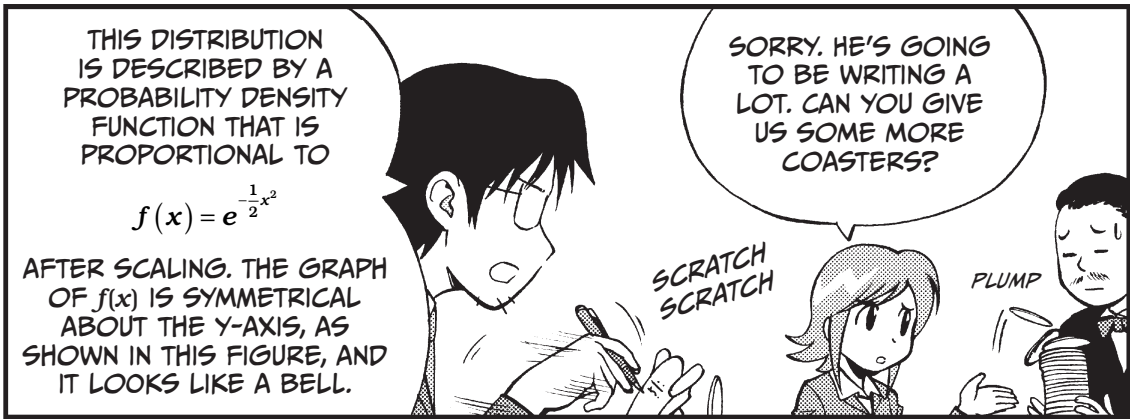
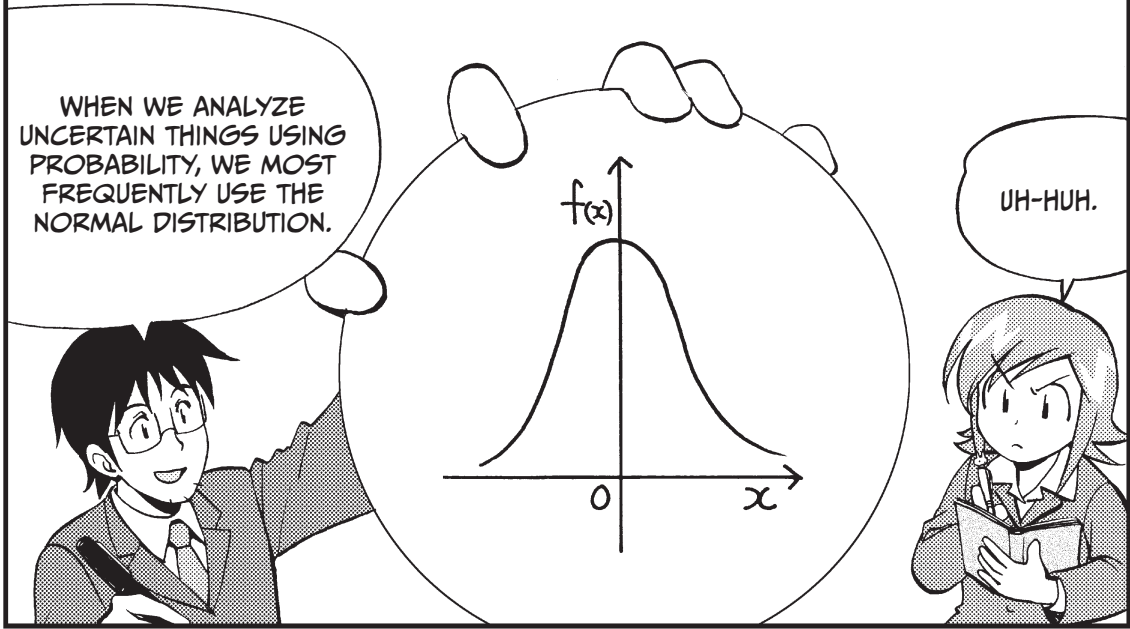


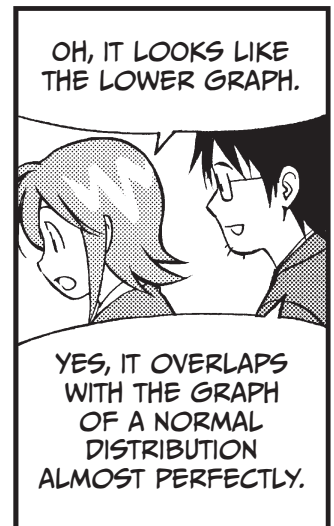
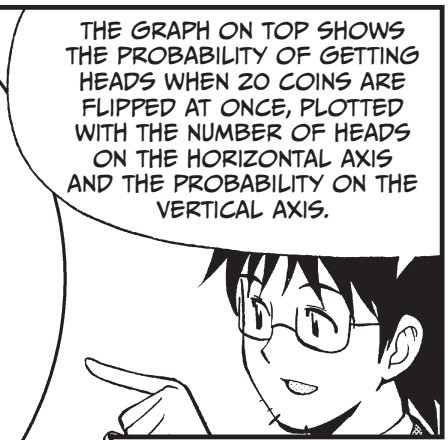
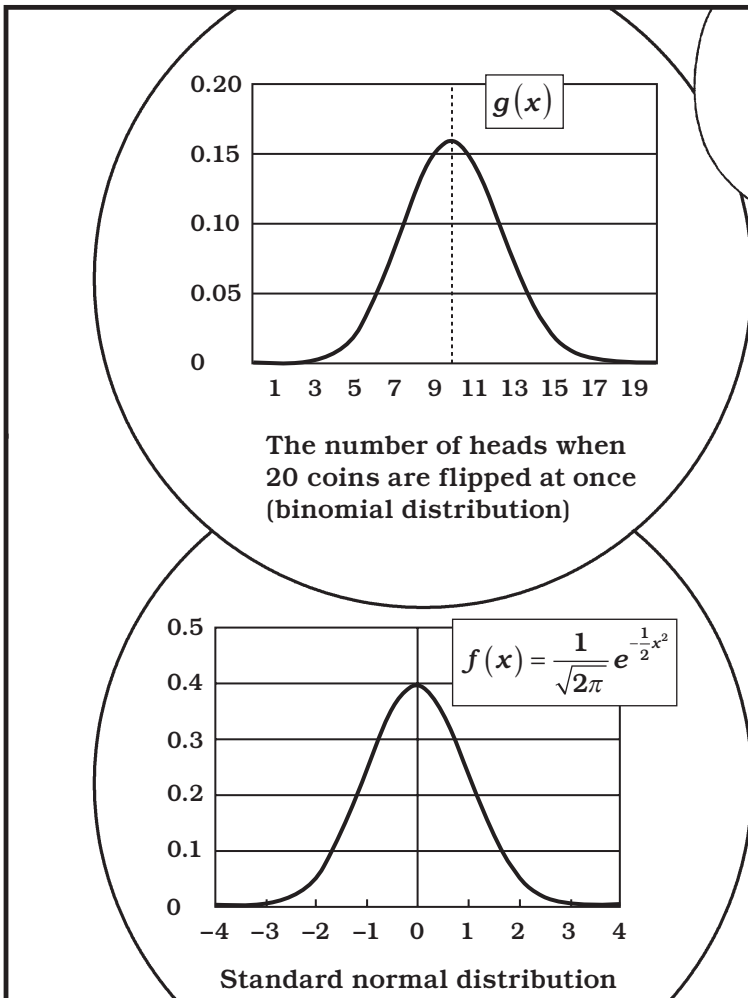
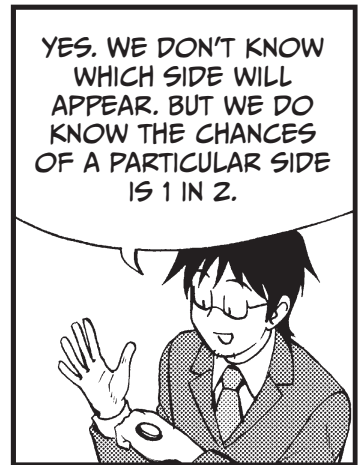
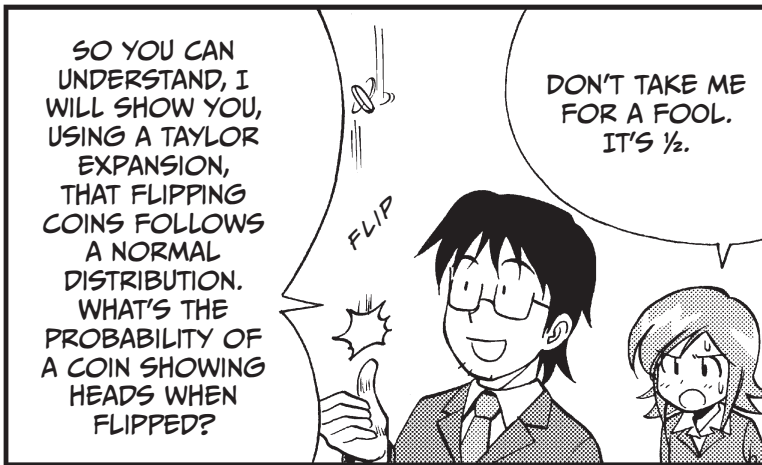
OH, SPEAKING OF PROBABILITY!

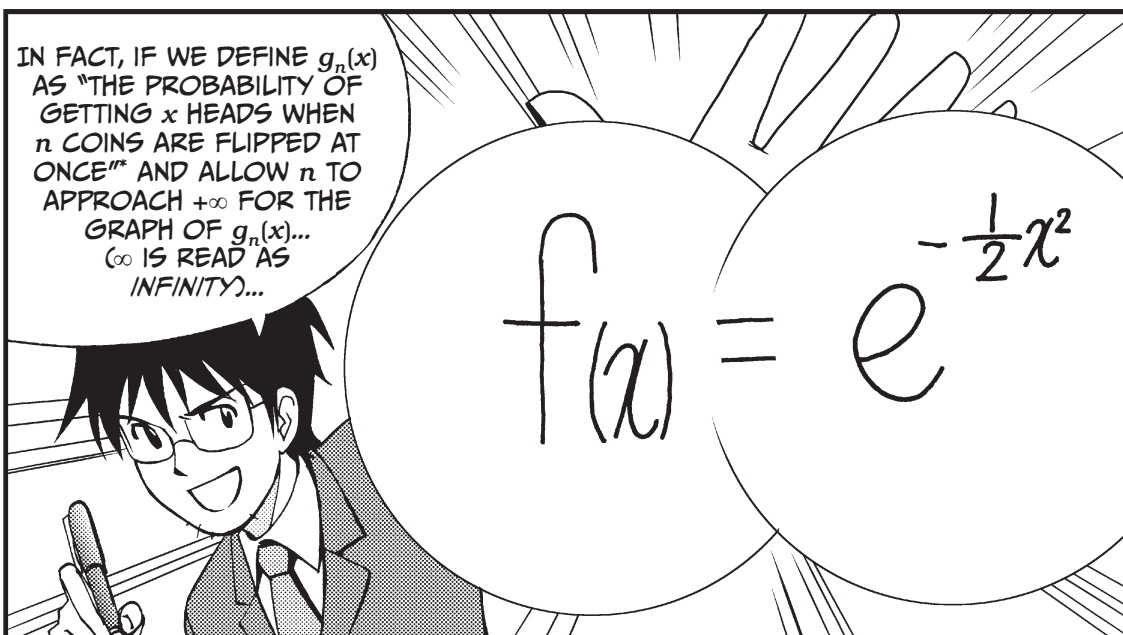
WHAT? NO WAY! WE'RE GOING TO STUDY NOW?



OF COURSE! I'M YOUR TEACHER, AND YOU ARE A PRECIOUS ASSET.







\* The distribution of such probabilities as that of getting  $x$  heads when  $n$  coins are flipped is generally called the *binomial distribution*.

For example, let's find the probability of getting 3 heads when 5 coins are flipped. The probability of getting HHTHT (H: heads, T: tails) is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^5$$

Since there are  ${}_5C_3$  ways of getting 3 heads and 2 tails, it is  ${}_5C_3 \left(\frac{1}{2}\right)^5$ . The general expression is  ${}_nC_x \left(\frac{1}{2}\right)^n$ . We will show that if  $n$  is very large, the binomial distribution is the normal distribution.

USING THE BINOMIAL DISTRIBUTION,

$$g_n(x) = {}_n C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}$$

$$= {}_n C_x \left(\frac{1}{2}\right)^n$$



$g_n(x)$  CAN BE WRITTEN IN THIS WAY.

SINCE THE GRAPH OF  $f(x)$  IS SYMMETRICAL ABOUT  $x = 0$  AND  $g_n(x)$  ABOUT  $x = \frac{1}{2}$  ...



AH, SO MANY COASTERS...

WE CONSIDER  $g_n\left(\frac{n}{2}\right)$  INSTEAD OF  $g_n(x)$ .

FIRST...

$$g_n\left(\frac{n}{2}\right) = {}_n C_{\frac{n}{2}} \left(\frac{1}{2}\right)^n$$

DIVIDING  $g_n(x)$  BY THIS...

$$h_n(x) = \frac{g_n(x)}{g_n\left(\frac{n}{2}\right)} = \frac{{}_n C_x}{{}_n C_{\frac{n}{2}}}$$

WE GET  $h_n$ , THE SCALED FUNCTION

SINCE

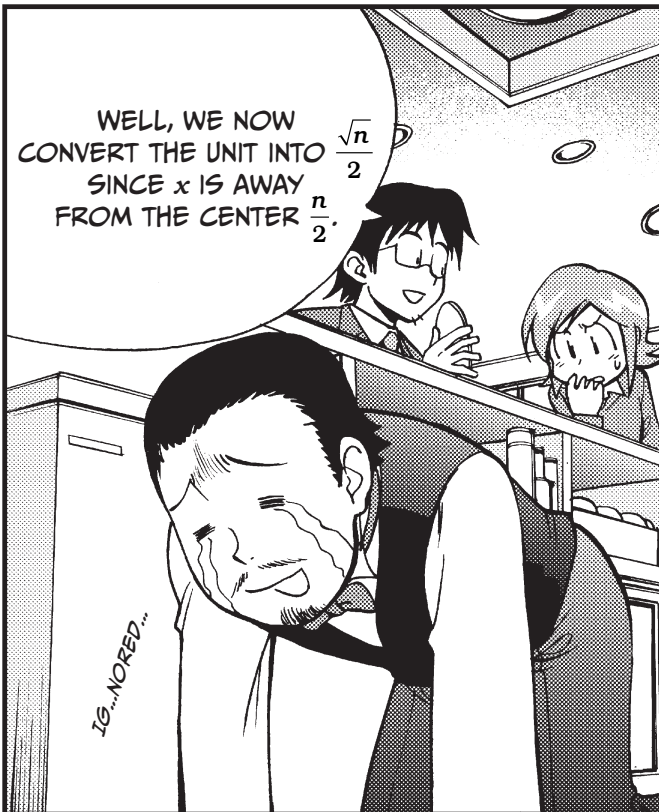
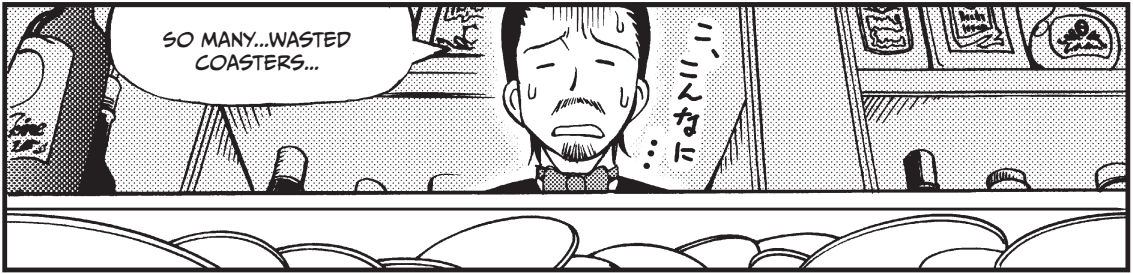
$${}_n C_x = \frac{n!}{x!(n-x)!}$$

SO THEN...

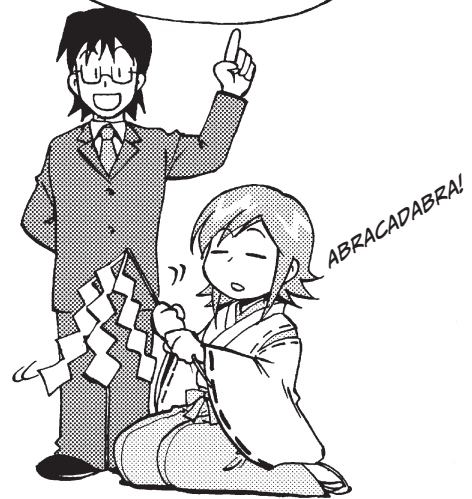
$${}_n C_{\frac{n}{2}} = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

DIVIDE:

$$h_n(x) = \left(\frac{n!}{x!(n-x)!}\right) \times \left(\frac{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\right) = \frac{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{x!(n-x)!}$$



$\frac{\sqrt{n}}{2}$  IS THE STANDARD DEVIATION. IF YOU DON'T KNOW STATISTICS, SIMPLY REGARD IT AS A MAGIC WORD!\*



\* STANDARD DEVIATION IS AN INDEX WE USE TO DESCRIBE THE SCATTERING OF DATA.

IN OTHER WORDS,

$$x = \frac{n}{2} + \frac{\sqrt{n}}{2} \times 1 \rightarrow z = 1$$

$$x = \frac{n}{2} + \frac{\sqrt{n}}{2} \times 2 \rightarrow z = 2$$

$$x = \frac{n}{2} + \frac{\sqrt{n}}{2} \times 3 \rightarrow z = 3$$

IN THIS WAY, WE CHANGE THE VARIABLE. THE NEW ONE,  $z$ , IS THE NUMBER OF STANDARD DEVIATIONS AWAY FROM THE CENTER.



WE SET  $\frac{n}{2} + \frac{\sqrt{n}}{2} z = x$   
AND SUBSTITUTE  $z$  IN  $h_n$ .

AND GET 
$$h_n(z) = \frac{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!}{\left(\frac{n}{2} + \frac{\sqrt{n}}{2} z\right)! \left(\frac{n}{2} - \frac{\sqrt{n}}{2} z\right)!}$$

$$\left[ n - \left( \frac{n}{2} + \frac{\sqrt{n}}{2} z \right) \right]$$

WE TAKE A  $\ln$  OF EACH SIDE.\*

$$\ln h_n(z)$$

$$= \ln \left[ \left( \frac{n}{2} \right)! \right] + \ln \left[ \left( \frac{n}{2} \right)! \right] - \ln \left[ \left( \frac{n}{2} + \frac{\sqrt{n}}{2} z \right)! \right] - \ln \left[ \left( \frac{n}{2} - \frac{\sqrt{n}}{2} z \right)! \right]$$

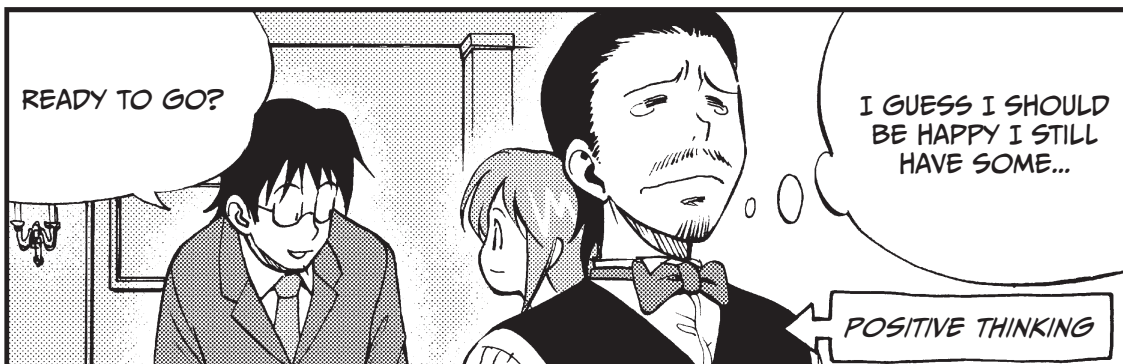
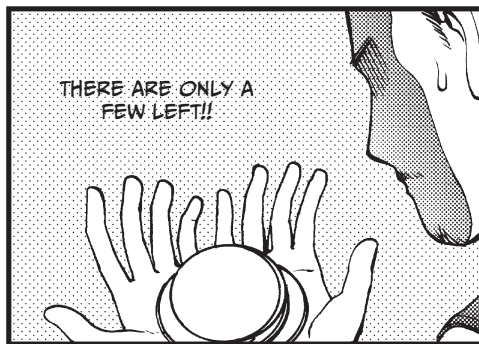
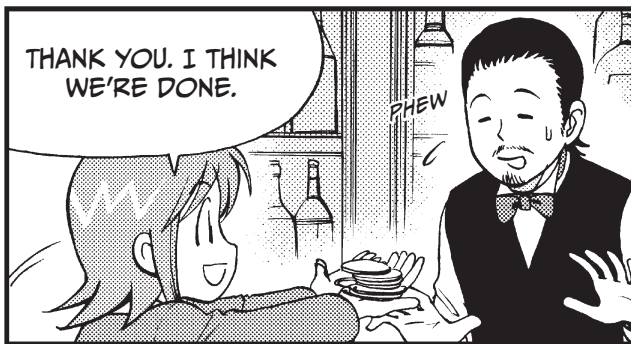
\* WE USE

$$\ln ab = \ln a + \ln b$$

$$\ln \frac{d}{c} = \ln d - \ln c$$

NOW WE NEED TO CALCULATE THIS, BUT SHALL WE MOVE ON TO THE NEXT BAR?





### Approximating $\ln(m!)$

$$\ln m! = \ln 1 + \ln 2 + \ln 3 + \dots + \ln m$$

If we pack rectangles in the graph of  $\ln x$ , as shown here, we get

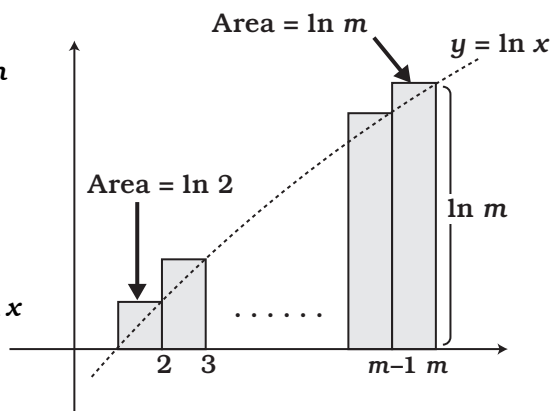
$$\ln 2 + \dots + \ln m \approx \int_1^m \ln x dx$$

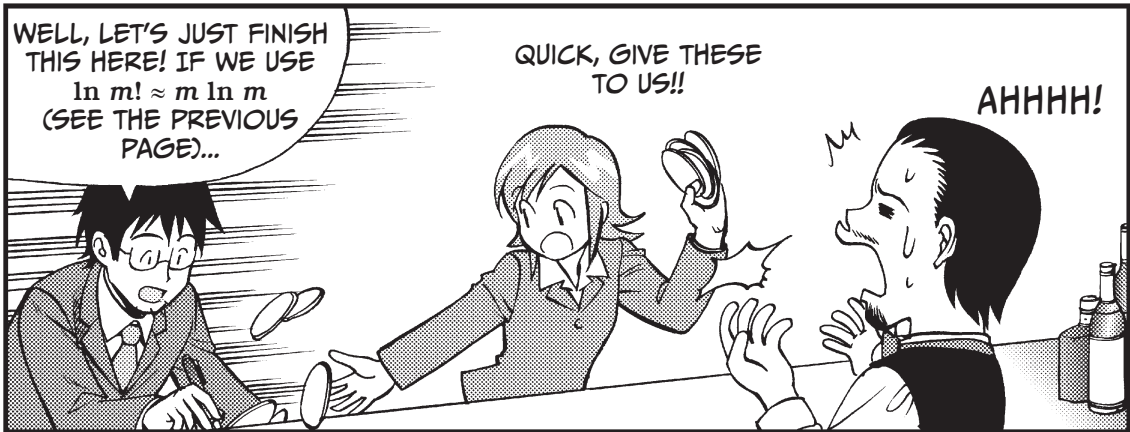
$$(x \ln x - x)' = \ln x + x \times \frac{1}{x} - 1 = \ln x$$

Thus,

$$\begin{aligned} \int_1^m \ln x dx &= (m \ln m - m) - (1 \ln 1 - 1) \\ &= m \ln m - m + 1 \end{aligned}$$

Since we will use this where  $m$  is very large,  $m \ln m$  is the important term.  $-m + 1$  is much smaller, so we'll ignore it. Therefore, we can consider roughly that  $\ln m! = m \ln m$ .






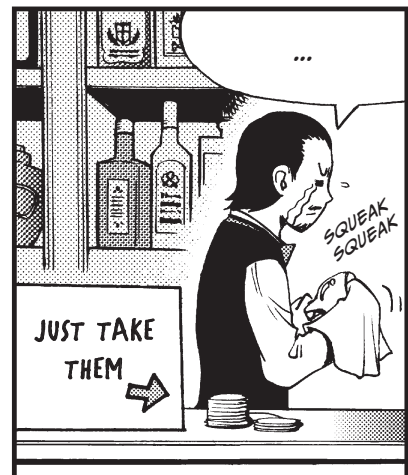
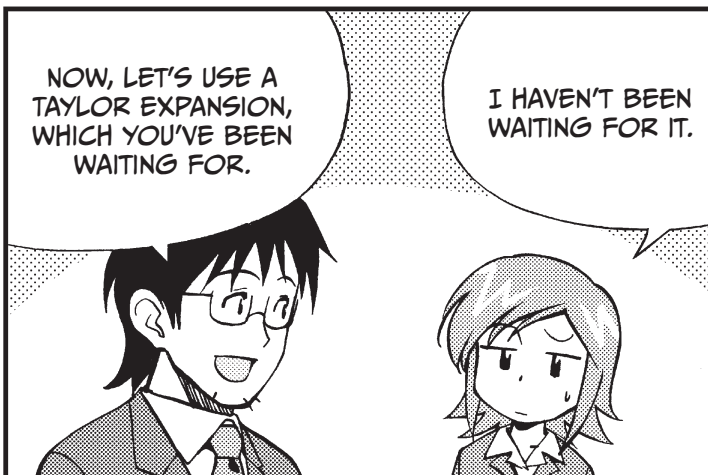
$$\ln h_n(x) \approx \frac{n}{2} \ln \frac{n}{2} + \frac{n}{2} \ln \frac{n}{2} - \left(\frac{n}{2} + \frac{\sqrt{n}}{2} z\right) \ln \left(\frac{n}{2} + \frac{\sqrt{n}}{2} z\right) - \left(\frac{n}{2} - \frac{\sqrt{n}}{2} z\right) \ln \left(\frac{n}{2} - \frac{\sqrt{n}}{2} z\right)$$

AFTER A LOT OF ALGEBRA, WE GET

$$\ln h_n(x) \approx -\left[\left(\frac{n}{2} + \frac{\sqrt{n}}{2} z\right) \ln \left(1 + \frac{\sqrt{n}}{n} z\right) + \left(\frac{n}{2} - \frac{\sqrt{n}}{2} z\right) \ln \left(1 - \frac{\sqrt{n}}{n} z\right)\right]$$

WE USED, E.G.,  $\ln \left(\frac{n}{2} + \frac{\sqrt{n}}{2} z\right) = \ln \left\{ \frac{n}{2} \left(1 + \frac{\sqrt{n}}{n} z\right)\right\} = \ln \frac{n}{2} + \ln \left(1 + \frac{\sqrt{n}}{n} z\right)$





WHEN  $t$  IS  
CLOSE TO  
ZERO,

$$\ln(1+t) \approx t - \frac{1}{2}t^2$$

(QUADRATIC  
APPROXIMATION)\*



NOW,  $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$  IS VERY  
CLOSE TO ZERO IF  $n$  IS  
LARGE ENOUGH.

$\frac{\sqrt{n}}{n}z$  ALSO IS THEREFORE  
AS CLOSE AS WE WANT  
TO ZERO FOR FIXED  $z$ .



\* SEE PAGE 161.

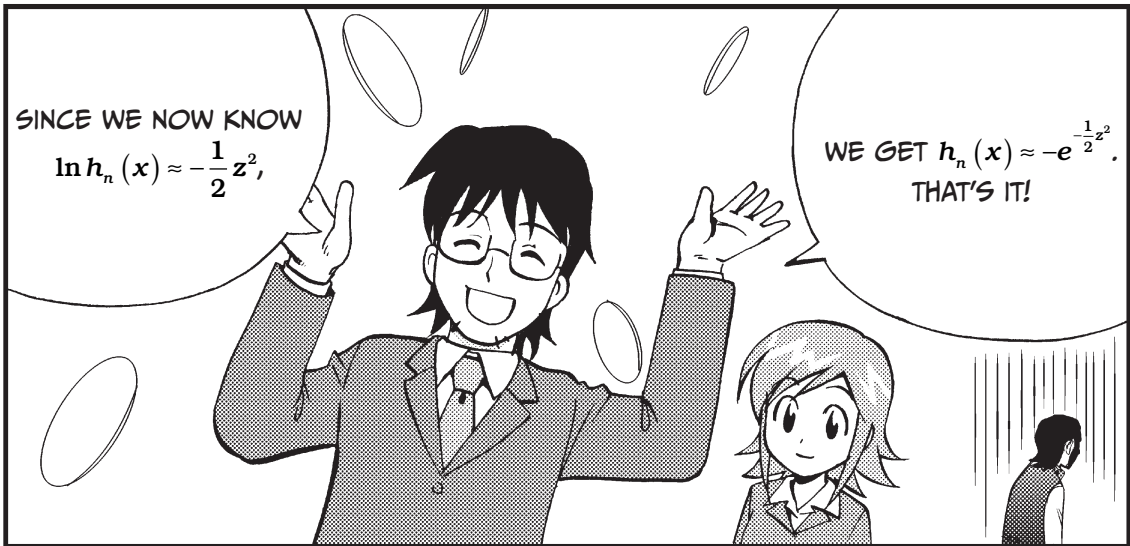
THEREFORE,

$$\ln\left(1 + \frac{\sqrt{n}}{n}z\right) \approx \frac{\sqrt{n}}{n}z - \frac{1}{2} \frac{1}{n}z^2$$

$$\ln\left(1 - \frac{\sqrt{n}}{n}z\right) \approx -\frac{\sqrt{n}}{n}z - \frac{1}{2} \frac{1}{n}z^2$$

WE PUT  
THESE BACK.

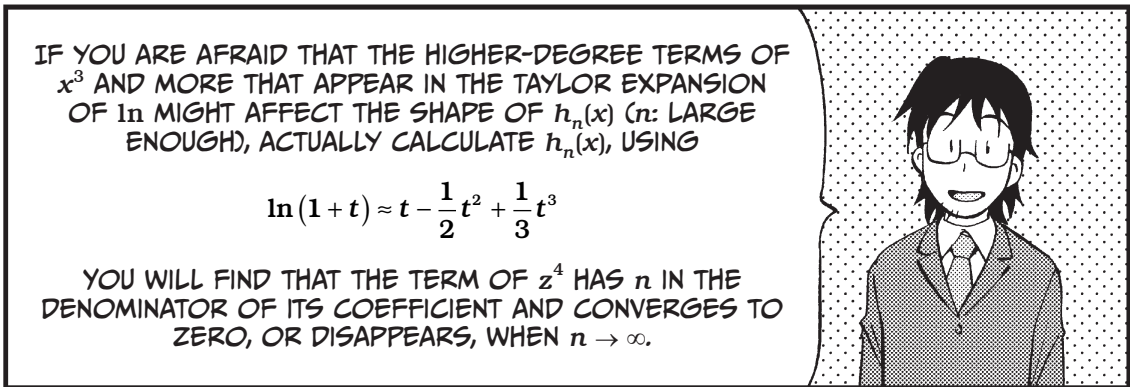
$$\begin{aligned} \ln h_n(x) &\approx -\left[\left(\frac{n}{2} + \frac{\sqrt{n}}{2}z\right)\left(\frac{\sqrt{n}}{n}z - \frac{1}{2} \frac{1}{n}z^2\right) + \left(\frac{n}{2} - \frac{\sqrt{n}}{2}z\right)\left(-\frac{\sqrt{n}}{n}z - \frac{1}{2} \frac{1}{n}z^2\right)\right] \\ &= -\left[z^2 - \frac{1}{2}z^2\right] = -\frac{1}{2}z^2 \end{aligned}$$



SINCE WE NOW KNOW

$$\ln h_n(x) \approx -\frac{1}{2}z^2,$$

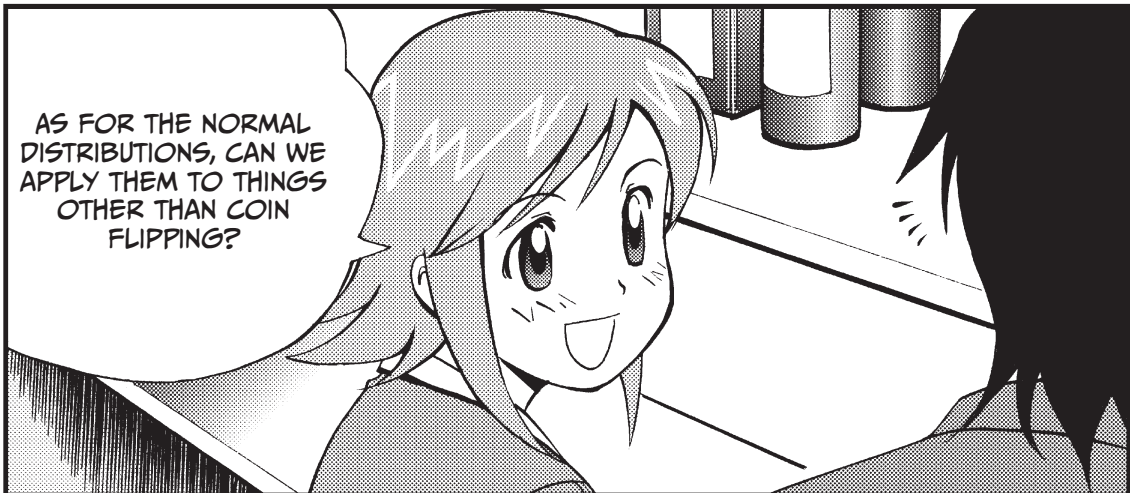
WE GET  $h_n(x) \approx -e^{-\frac{1}{2}z^2}$ .  
THAT'S IT!



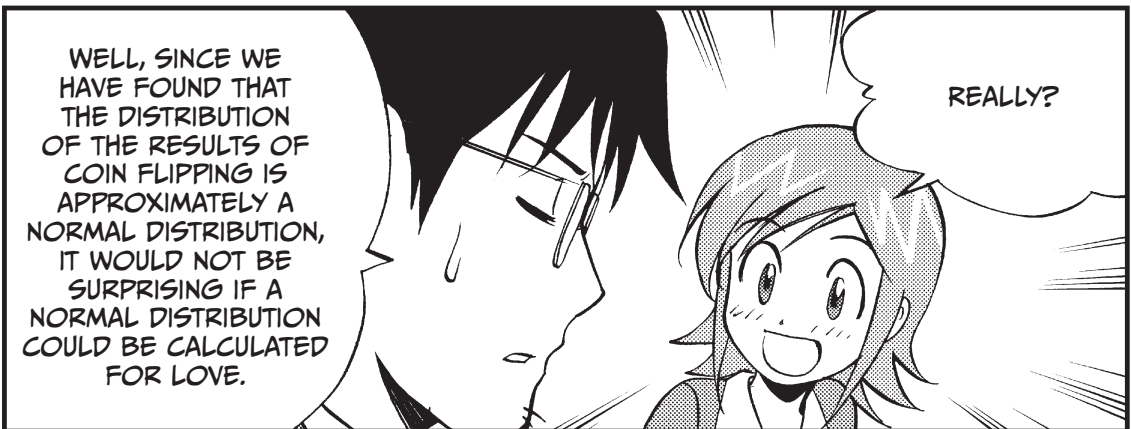
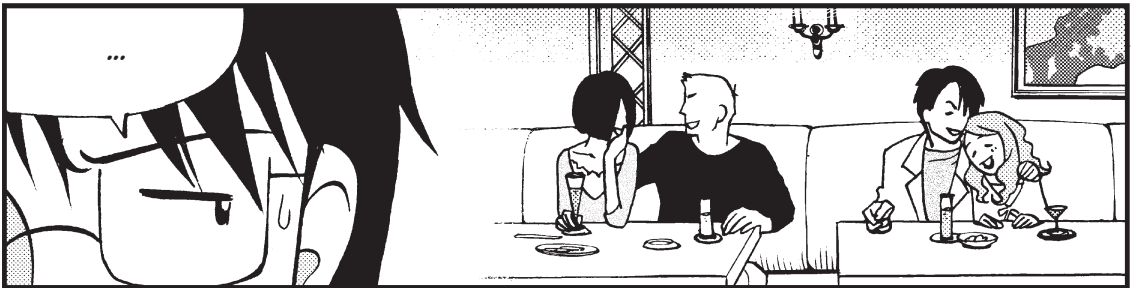
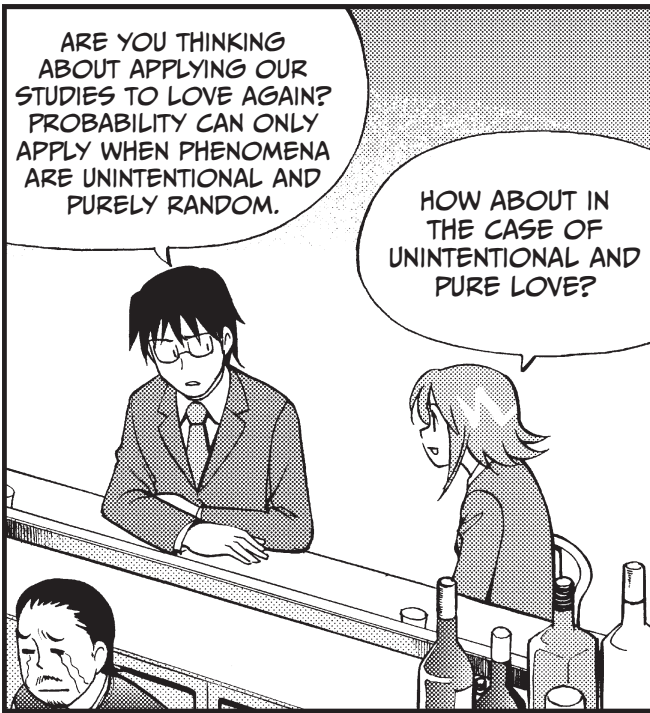
IF YOU ARE AFRAID THAT THE HIGHER-DEGREE TERMS OF  $x^3$  AND MORE THAT APPEAR IN THE TAYLOR EXPANSION OF  $\ln$  MIGHT AFFECT THE SHAPE OF  $h_n(x)$  ( $n$ : LARGE ENOUGH), ACTUALLY CALCULATE  $h_n(x)$ , USING

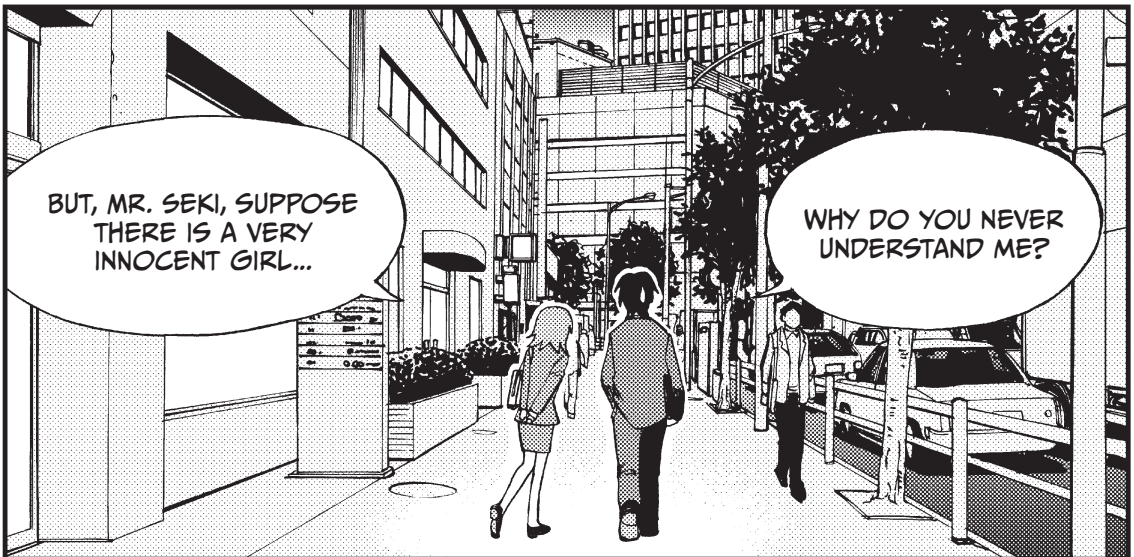
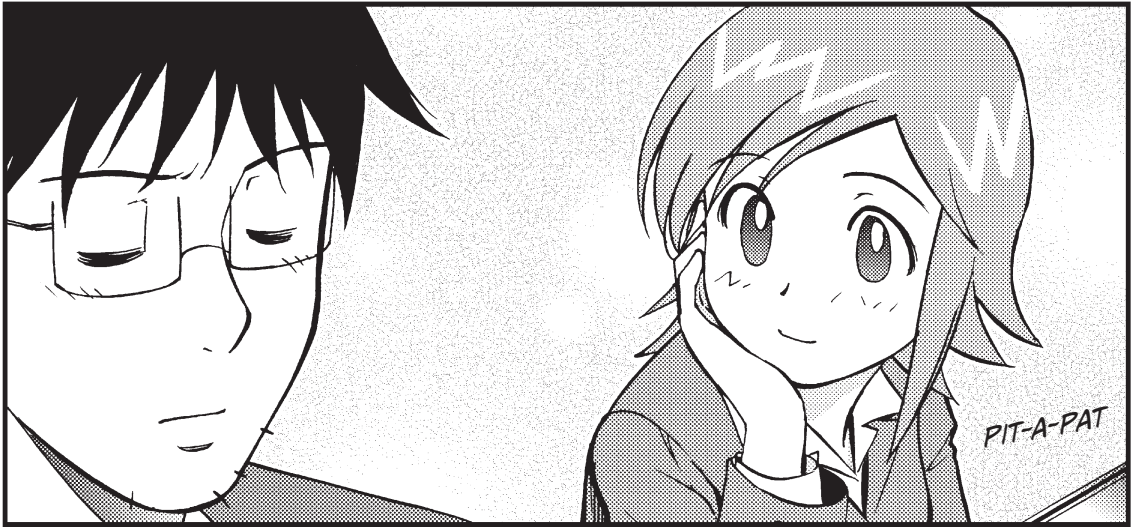
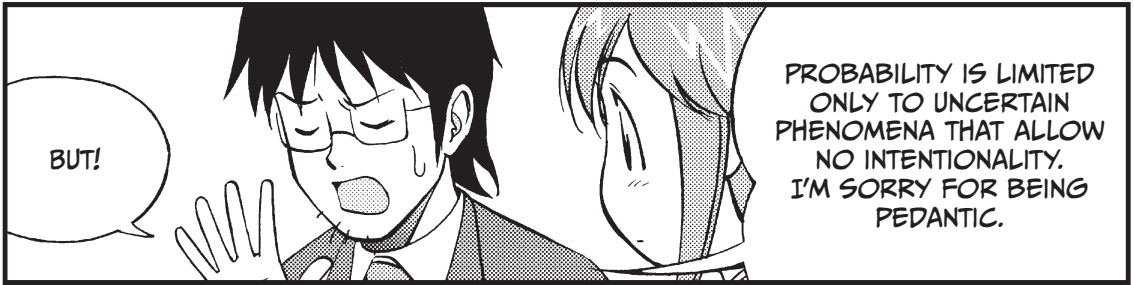
$$\ln(1+t) \approx t - \frac{1}{2}t^2 + \frac{1}{3}t^3$$

YOU WILL FIND THAT THE TERM OF  $z^4$  HAS  $n$  IN THE DENOMINATOR OF ITS COEFFICIENT AND CONVERGES TO ZERO, OR DISAPPEARS, WHEN  $n \rightarrow \infty$ .



AS FOR THE NORMAL DISTRIBUTIONS, CAN WE APPLY THEM TO THINGS OTHER THAN COIN FLIPPING?





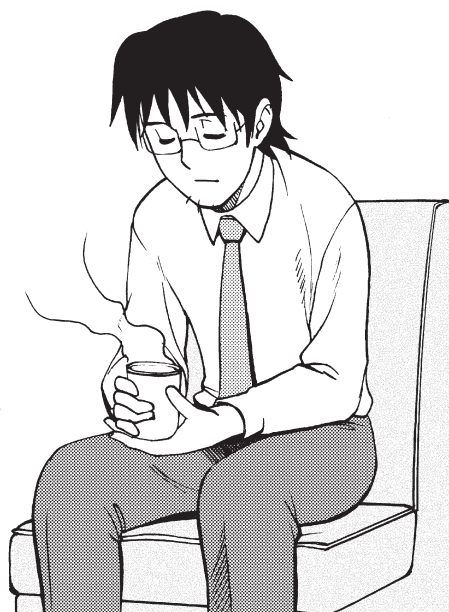
## EXERCISES

1. Obtain the Taylor expansion of  $f(x) = e^{-x}$  at  $x = 0$ .
2. Obtain the quadratic approximation of  $f(x) = \frac{1}{\cos x}$  at  $x = 0$ .
3. Derive for yourself the formula for the Taylor expansion of  $f(x)$  centered at  $x = a$ , which is given on page 159. In other words, work out what  $c_n$  must be in the equation:

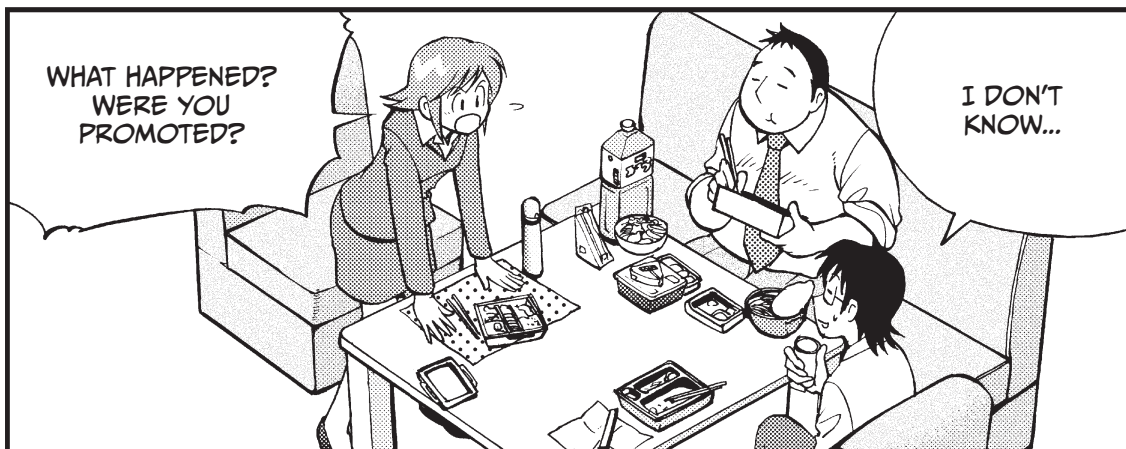
$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n$$

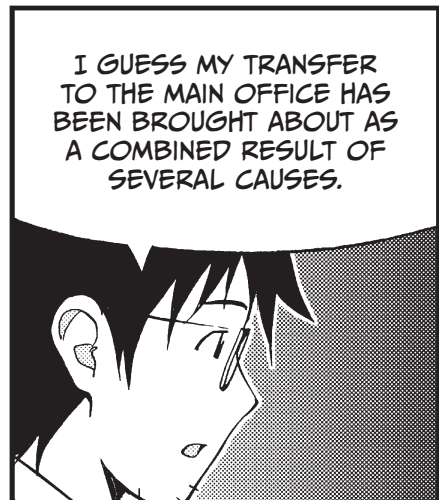
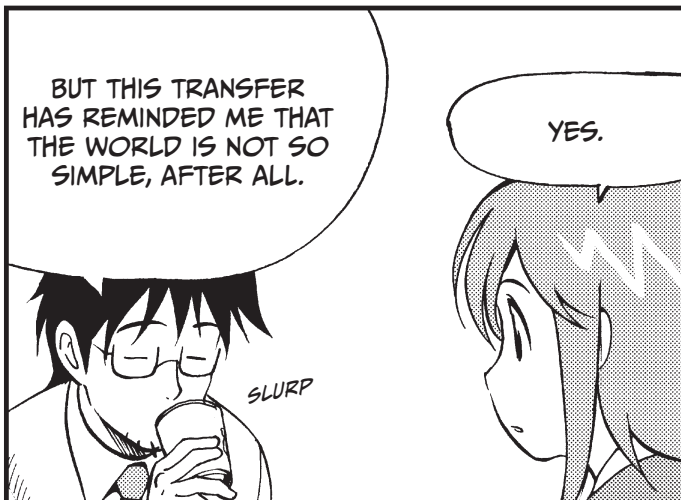
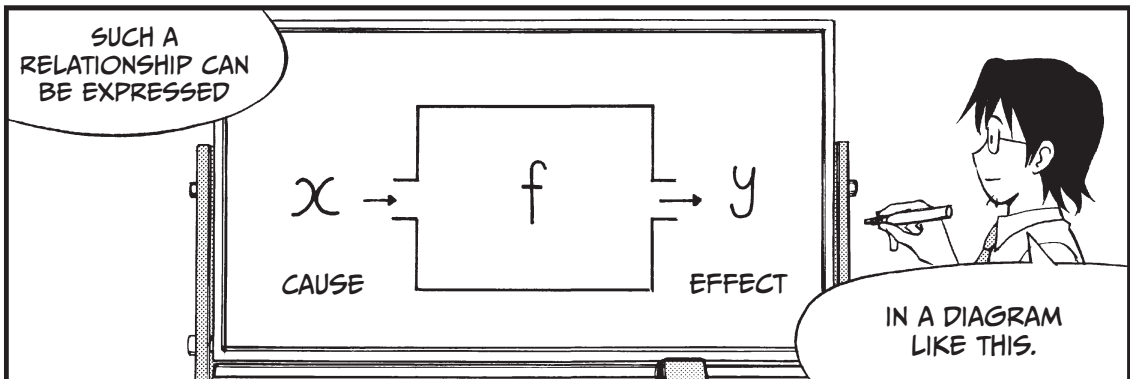
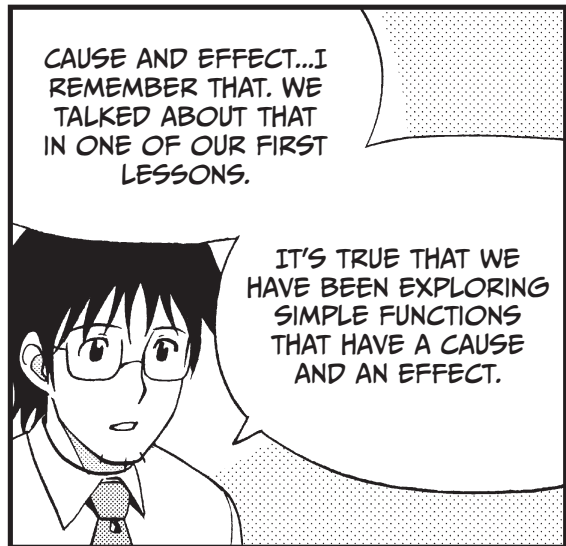
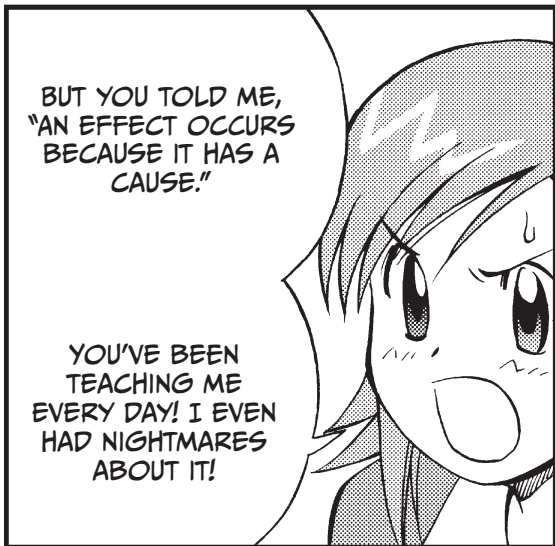
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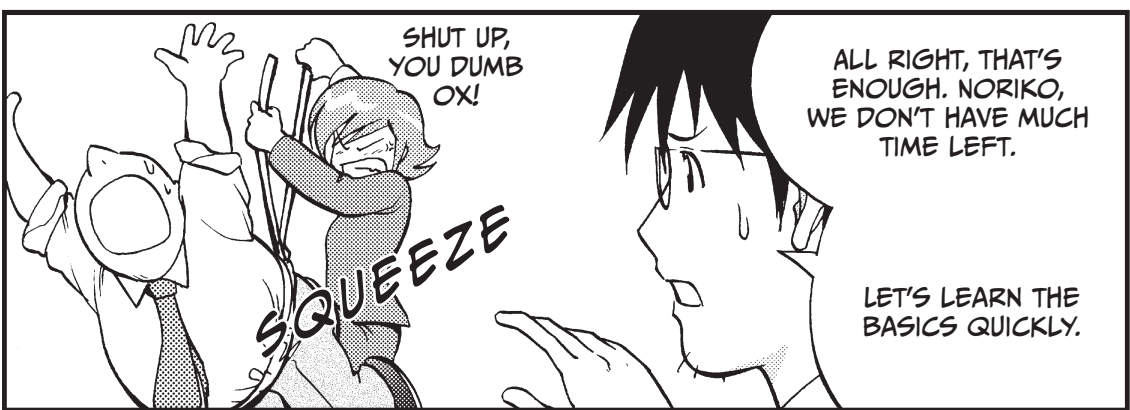
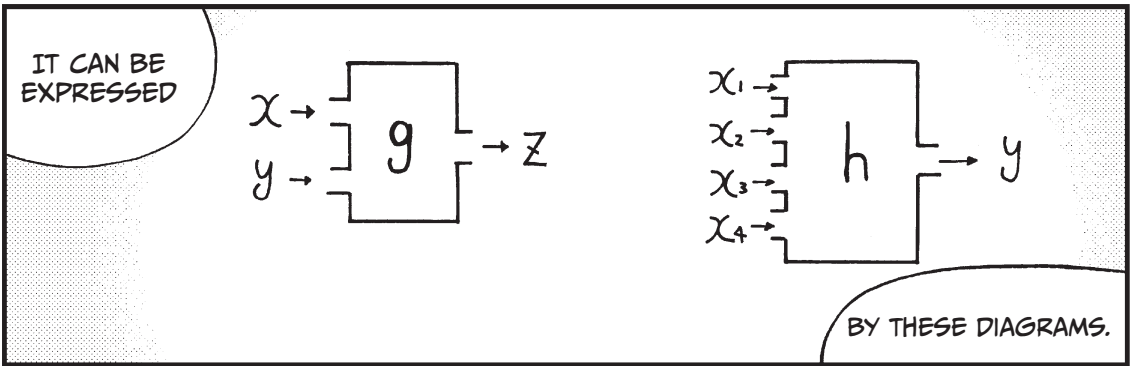
LET'S LEARN ABOUT  
PARTIAL DIFFERENTIATION!

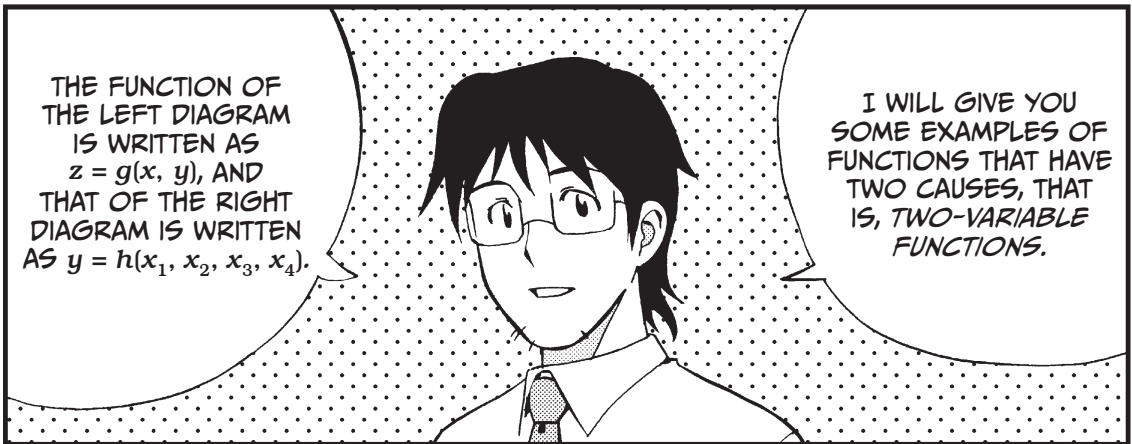


# WHAT ARE MULTIVARIABLE FUNCTIONS?









**EXAMPLE 1**

Assume that an object is at height  $h(v, t)$  in meters after  $t$  seconds when it is thrown vertically upward from the ground with velocity  $v$ . Then,  $h(v, t)$  is given by

$$h(v, t) = vt - 4.9t^2$$

**EXAMPLE 2**

The concentration  $f(x, y)$  of sugar syrup obtained by dissolving  $y$  grams of sugar in  $x$  grams of water is given by

$$f(x, y) = \frac{y}{x + y} \square 100$$

**EXAMPLE 3**

When the amount of equipment and machinery (called *capital*) in a nation is represented with  $K$  and the amount of labor by  $L$ , we assume that the total production of commodities (GDP: Gross Domestic Product) is given by  $Y(L, K)$ .



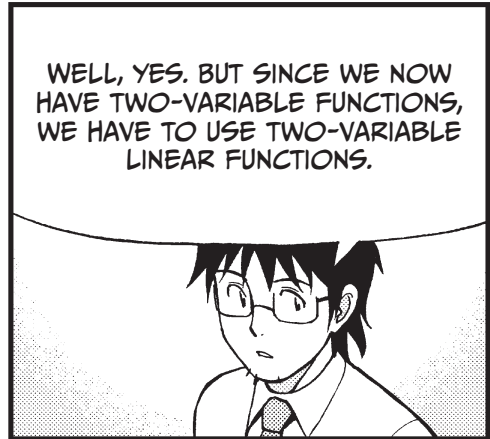
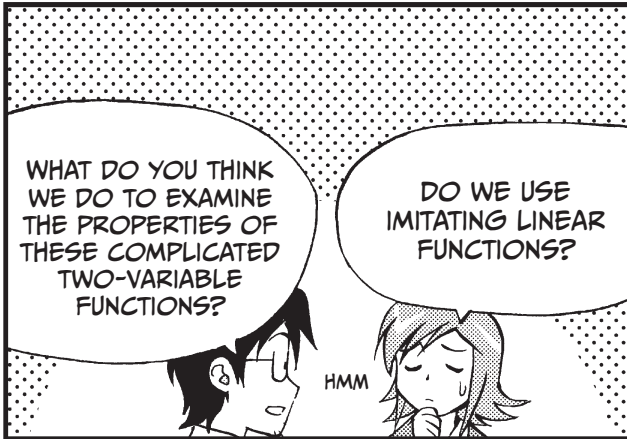
IN ECONOMICS,  $Y(L, K) = \beta L^\alpha K^{1-\alpha}$  (CALLED THE COBB-DOUGLAS FUNCTION) (WHERE  $\alpha$  AND  $\beta$  ARE CONSTANTS) IS USED AS AN APPROXIMATE FUNCTION OF  $Y(L, K)$ . SEE PAGE 203.

**EXAMPLE 4**

In physics, when the pressure of an ideal gas is given by  $P$  and its volume by  $V$ , its temperature  $T$  is known to be a function of  $P$  and  $V$  as  $T(P, V)$ . And it is given by

$$T(P, V) = \gamma PV$$

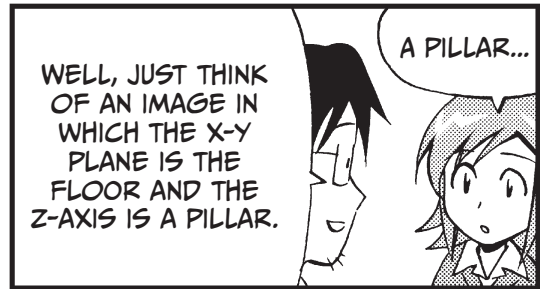
# THE BASICS OF VARIABLE LINEAR FUNCTIONS

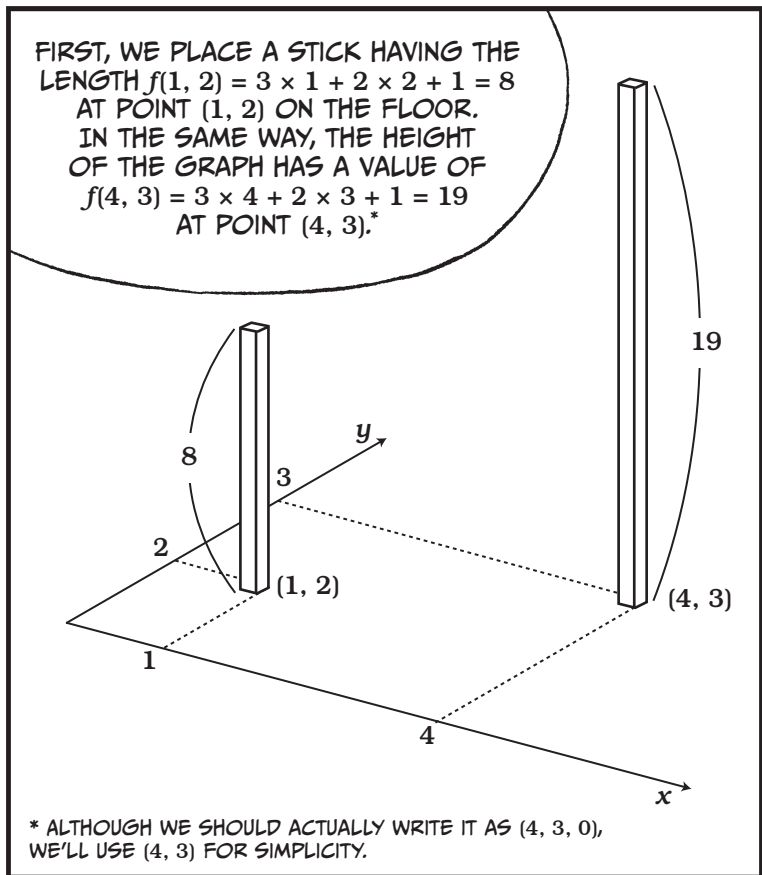
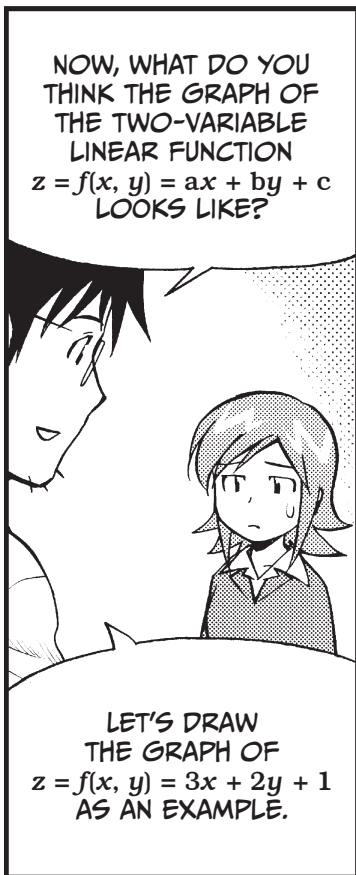
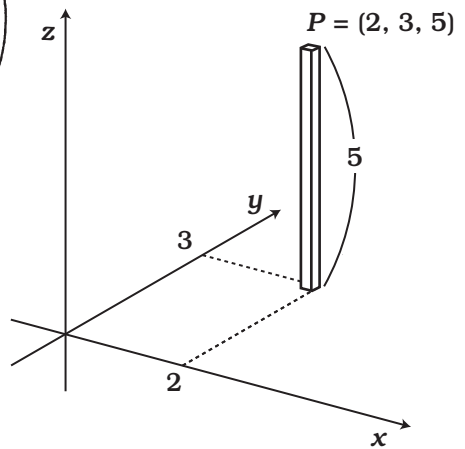
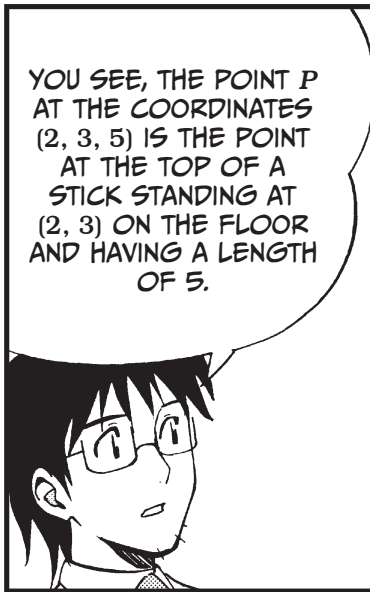
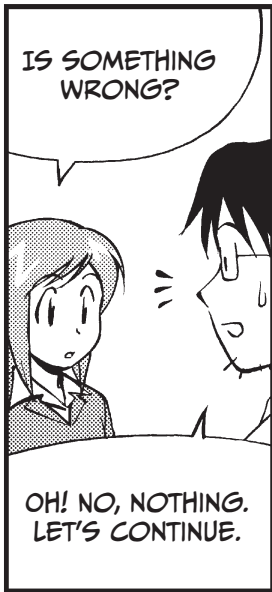


TWO-VARIABLE LINEAR FUNCTIONS ARE GIVEN IN A FORM LIKE  $z = f(x, y) = ax + by + c$  (WHERE  $a$ ,  $b$ , AND  $c$  ARE CONSTANTS.)

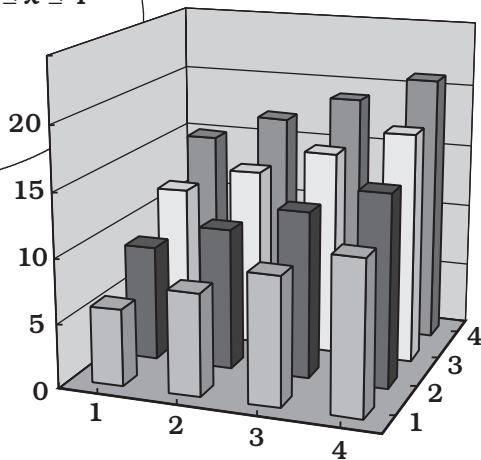
$x$  →  $y$  →  $f$  →  $z = ax + by + c$

FOR EXAMPLE,  
 $z = 3x + 2y + 1$  OR  
 $z = -x + 9y - 2$ ,  
SEE?

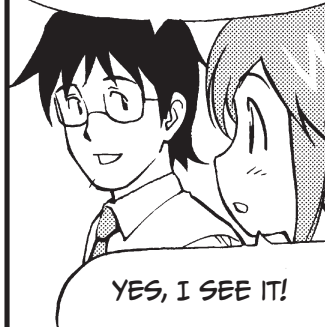




IN THE SAME WAY, WE PUT UP 16 STICKS AT 16 POINTS  $(x, y)$  SATISFYING  $1 \leq x \leq 4$  AND  $1 \leq y \leq 4$ , WHICH ARE SHOWN IN THIS FIGURE.



LOOKING AT THIS FIGURE, YOU CAN VAGUELY SEE THAT THE GRAPH FORMS A PLANE, CAN'T YOU?



NOW, LET'S LOOK AT THE PILLARS ON THE NEAREST SIDE.

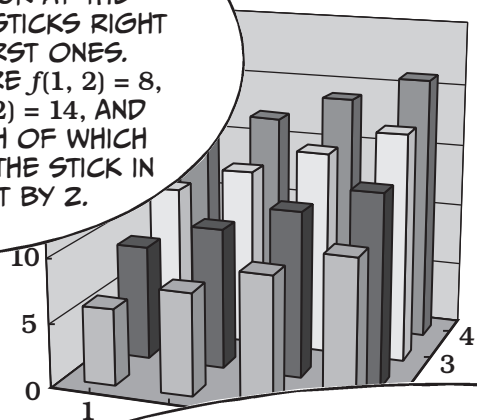


THEIR HEIGHTS ARE, BEGINNING FROM THE LEFT,  $f(1, 1) = 6$ ,  $f(2, 1) = 9$ ,  $f(3, 1) = 12$ , AND  $f(4, 1) = 15$ .

THESE POINTS FORM A LINE WHOSE SLOPE IS 3, WHICH IS INTUITIVE BECAUSE IF  $y$  IS A CONSTANT ( $y = 1$ ) IN  $z = f(x, y) = 3x + 2y + 1$ , WE GET  $z = 3x + 2 \times 1 + 1 = 3x + 3$ .

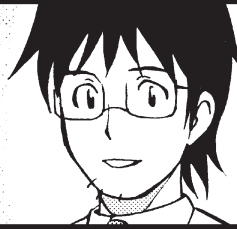


NEXT, LET'S LOOK AT THE HEIGHTS OF THE STICKS RIGHT BEHIND THE FIRST ONES. THEIR HEIGHTS ARE  $f(1, 2) = 8$ ,  $f(2, 2) = 11$ ,  $f(3, 2) = 14$ , AND  $f(4, 2) = 17$ , EACH OF WHICH IS HIGHER THAN THE STICK IN FRONT OF IT BY 2.



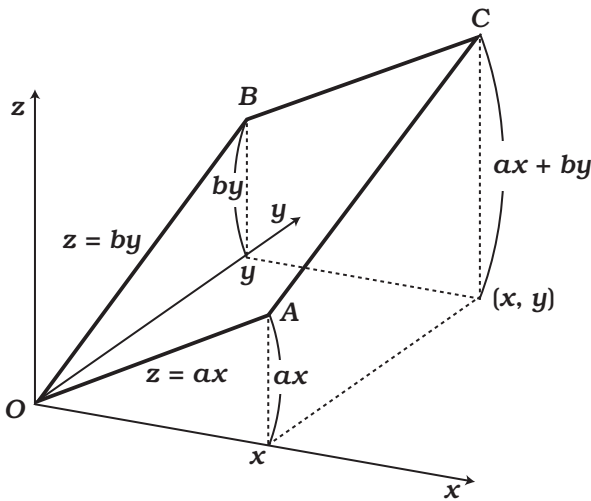
FURTHERMORE, THE HEIGHTS OF THE STICKS BEHIND THESE ONES ARE  $f(1, 3) = 10$ ,  $f(2, 3) = 13$ ,  $f(3, 3) = 16$ , AND  $f(4, 3) = 19$ , EACH OF WHICH IS AGAIN HIGHER THAN THE ONE IN FRONT OF IT BY 2.

SINCE THE STICKS BECOME HIGHER BY  $z$  THE FURTHER AWAY FROM US THEY ARE,



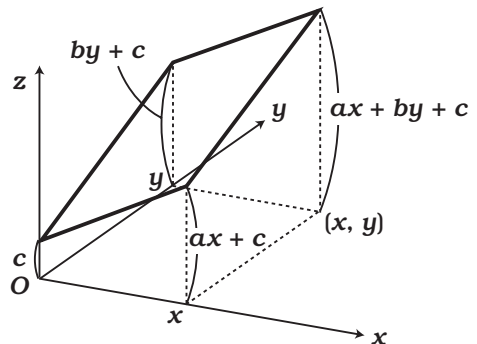
WE FIND THAT THE TOPS OF THE STICKS AS A WHOLE FORM A PLANE. WE CAN NOW GENERALIZE THIS.

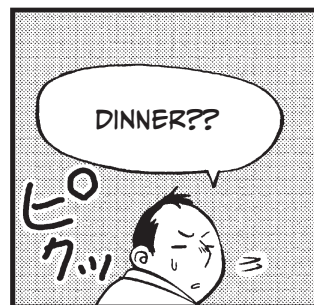
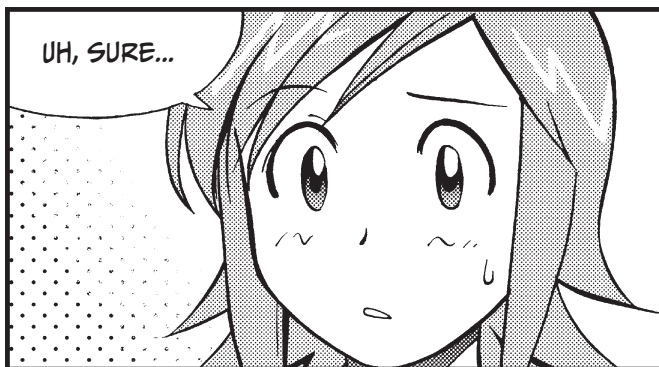
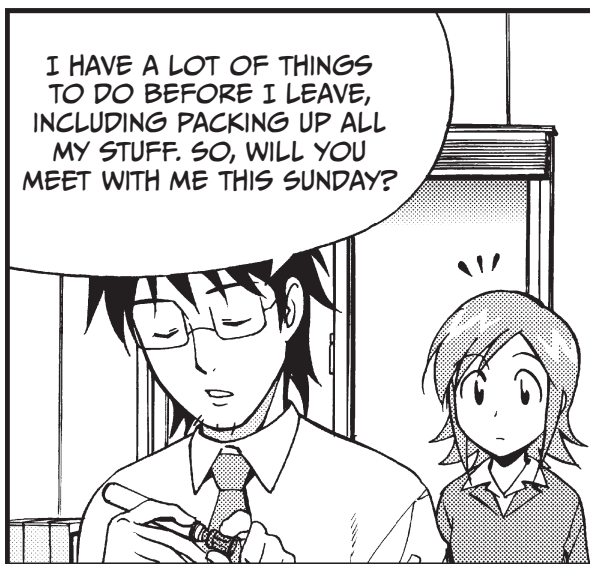
FIRST, LET'S DRAW THE GRAPH OF  $z = f(x, y) = ax + by$  (LET CONSTANT  $c = 0$ ).



LET'S CONSIDER A PLANE THAT REPRESENTS THE FUNCTION  $f(x, y)$ . WE CAN START AT POINT O, WHICH WE KNOW IS  $(0, 0, 0)$ , OR THE ORIGIN. NOW CONSIDER LINE SEGMENT OA—A FUNCTION TO DESCRIBE THIS LINE CAN BE FOUND IF WE SET  $y = 0$ . THIS MEANS THAT LINE IS REPRESENTED BY THE FUNCTION  $z = ax$ , AND HAS SLOPE  $a$ . SIMILARLY, WE FIND THAT LINE SEGMENT OB OF THIS PLANE IS REPRESENTED BY THE FUNCTION  $z = by$  (AS WE HAVE SET  $x$  EQUAL TO ZERO), AND HAS A SLOPE OF  $b$ . POINT C ON THE PLANE OACB HAS A HEIGHT EQUAL TO  $ax + by$ . IF WE WANTED TO PHYSICALLY REPRESENT THIS PLANE, WE COULD TIE A SHEET TO LINE SEGMENTS OA AND OB, AND TIGHTEN THE SHEET.

NOW, IF WE HAVE TO CONSIDER A CONSTANT (AN EQUATION THAT TAKES THE FORM  $z = ax + by + c$ ) WE SIMPLY ADJUST THE GRAPH BY RAISING THE PLANE BY  $c$ . POINT O ON OUR PLANE IS NOW AT  $(0, 0, c)$ , POINT A HAS A HEIGHT OF  $(ax + c)$ , AND SO ON.





TANAKA SCHOOL  
SUNDAY

山  
中  
学  
校

THIS SCHOOL WAS  
CLOSED A FEW  
YEARS AGO.

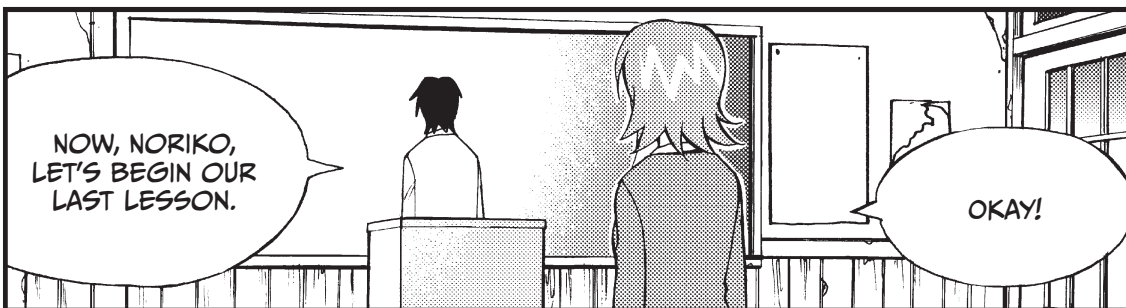
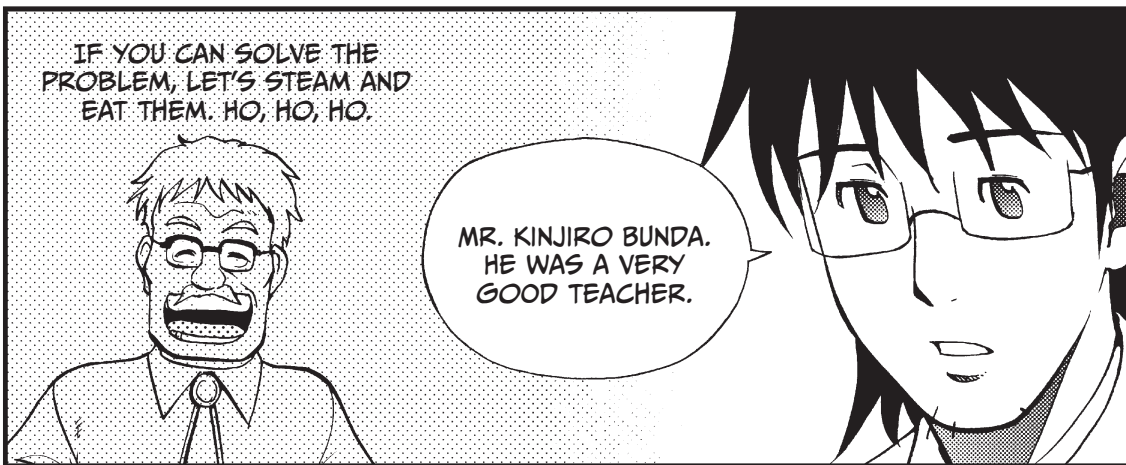
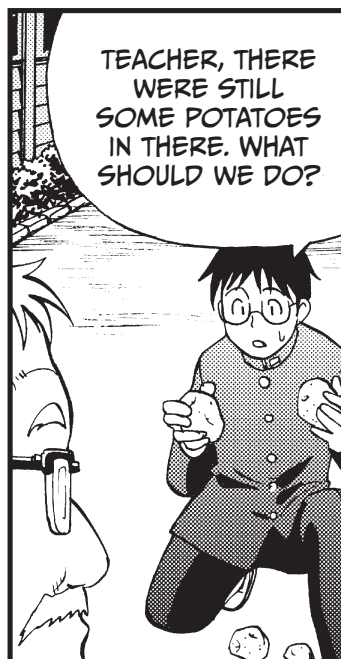
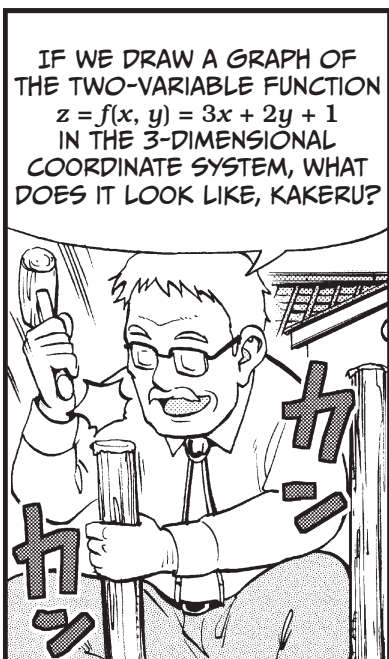
NO. I JUST LIKE IT  
HERE BECAUSE IT'S  
WHERE I LEARNED  
MATH.

REALLY? ARE YOU  
GOING TO WRITE A  
STORY ABOUT IT?

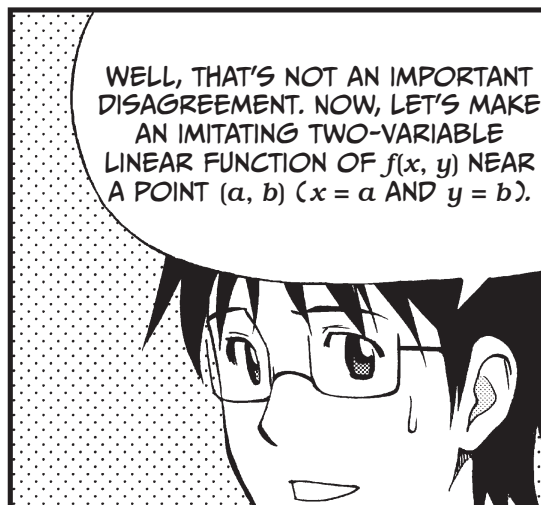
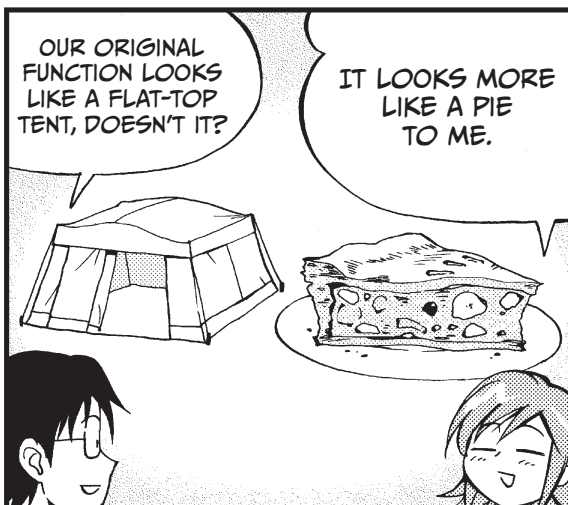
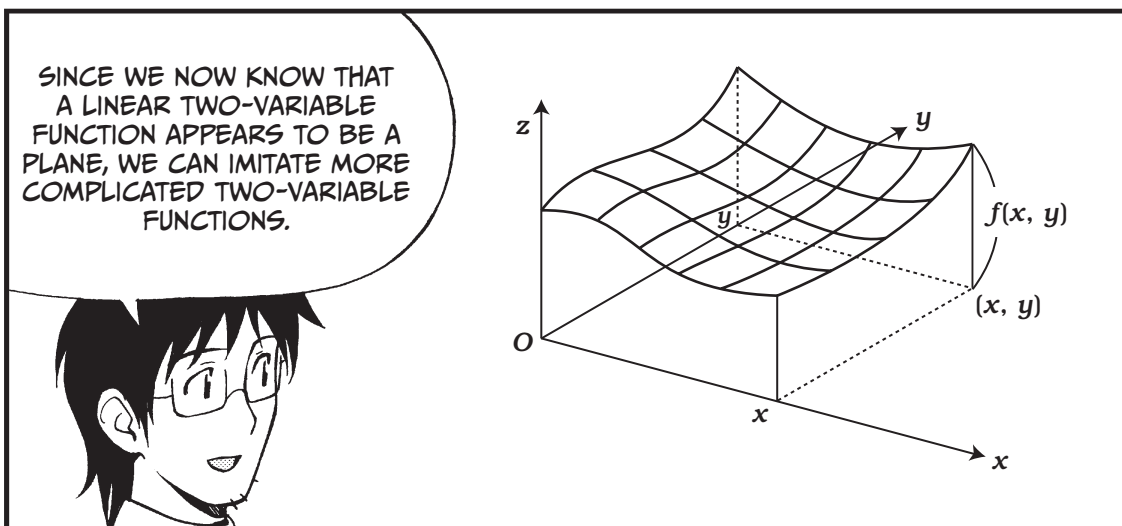
WOW!

ACTUALLY, I WAS  
BORN IN THIS TOWN.

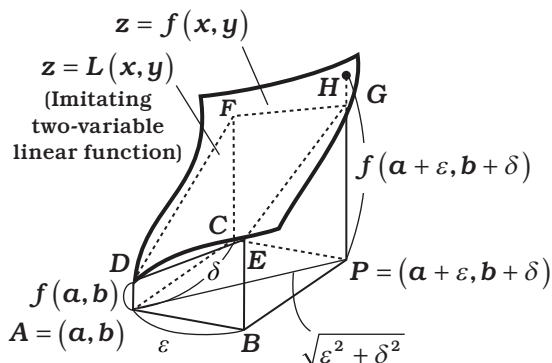
THIS WAS A SMALL SCHOOL.  
BUT THERE WAS A TEACHER  
HERE WHO GAVE ME THE BEST  
LESSONS IN THE WORLD.



# PARTIAL DIFFERENTIATION



We make a two-variable linear function that has the same height as  $f(a, b)$  at the point  $(a, b)$ . The formula is  $L(x, y) = p(x - a) + q(y - b) + f(a, b)$ . Substituting  $a$  for  $x$  and  $b$  for  $y$ , we get  $L(a, b) = f(a, b)$ .



While the graph of  $z = f(x, y)$  and that of  $z = L(x, y)$  pass through the same point above the point  $A = (a, b)$ , they differ in height at the point  $P = (a + \varepsilon, b + \delta)$ . The error in this case is  $f(a + \varepsilon, b + \delta) - L(a + \varepsilon, b + \delta) = f(a + \varepsilon, b + \delta) - f(a, b) - (p\varepsilon + q\delta)$ , and the relative error expresses the ratio of the error to the distance  $AP$ .

$$\text{Relative error} = \frac{\text{difference between } f \text{ and } L}{\text{distance } AP}$$

$$\textcircled{1} = \frac{f(a + \varepsilon, b + \delta) - f(a, b) - (p\varepsilon + q\delta)}{\sqrt{\varepsilon^2 + \delta^2}}$$

We consider  $L(x, y)$  as the difference between it and  $f$  becomes infinitely close to zero (when  $P$  is infinitely close to  $A$ ) as the imitating linear function. For that case, we obtain  $p$  and  $q$ .  $p$  is the slope of  $DE$  and  $q$  that of  $DF$  in the figure. Since  $\varepsilon$  and  $\delta$  are arbitrary, we first let  $\delta = 0$  and analyze  $\textcircled{1}$ .

$\textcircled{1}$  becomes

$$\begin{aligned} \text{Relative error} &= \frac{f(a + \varepsilon, b + 0) - f(a, b) - (p\varepsilon + q \times 0)}{\sqrt{\varepsilon^2 + 0^2}} \\ &= \frac{f(a + \varepsilon, b) - f(a, b)}{\varepsilon} - p \end{aligned}$$

Thus, the statement “the relative error  $\rightarrow 0$  when  $\varepsilon \rightarrow 0$ ” means the following:

$$\textcircled{2} \quad \lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon, b) - f(a, b)}{\varepsilon} = p$$

This is the slope of  $DE$ .

Here, we should realize that the left side of this expression is the same as single-variable differentiation. In other words, if we substitute  $b$  for  $y$  and keep it constant, we obtain  $f(x, b)$ , which is a function of  $x$  only. The left side of  $\textcircled{2}$  is then the calculation of finding the derivative of this function at  $x = a$ .

Although we are very much tempted to write the left side as  $f'(a, b)$  since it is a derivative, it would then be impossible to tell with respect to which,  $x$  or  $y$ , we differentiated it.

So, we write “the derivative of  $f$  obtained at  $x = a$  while  $y$  is fixed at  $b$ ” as  $f_x(a, b)$ .

This  $f_x$  is called “the partial derivative of  $f$  in the direction of  $x$ ”. This is the notation corresponding to the “prime” in single-variable differentiation.

The notation  $\frac{df}{dx}(a, b)$ , that corresponds to  $\frac{\partial f}{\partial x}$ , is also used. In short, we have the following:

“The derivative of  $f$  in the direction of  $x$  obtained at  $x = a$  while  $y$  is fixed at  $b$ ”

$$f_x(a, b) = \frac{\partial f}{\partial x}(a, b) \quad \text{also written as} \quad \left( \left[ \frac{\partial f}{\partial x} \right]_{x=a, y=b} \right)$$

= Slope of  $DE$

$\partial$  IS READ AS  
“PARTIAL DERIVATIVE.”

In exactly the same way, we can obtain the following.

“The derivative of  $f$  in the direction of  $y$  obtained at  $y = b$  while  $x$  is fixed at  $a$ ”

$$f_y(a, b) = \frac{\partial f}{\partial y}(a, b)$$

= Slope of  $DF$



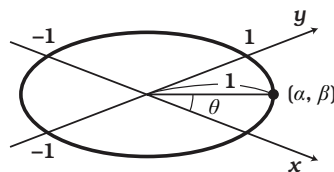
We have now found the following.

If  $z = f(x, y)$  has an imitating linear function near  $(x, y) = (a, b)$ , it is given by

$$\textcircled{3} \quad z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

$$\text{or}^* \quad z = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + f(a, b)$$

Consider a point  $(\alpha, \beta)$  on a circle with radius 1 centered at the origin of the  $x - y$  plane (the floor). We have  $\alpha^2 + \beta^2 = 1$  (or  $\alpha = \cos \theta$  and  $\beta = \sin \theta$ ). We now calculate the derivative in the direction from  $(0, 0)$  to  $(\alpha, \beta)$ . A displacement of distance  $t$  in this direction is expressed as  $(a, b) \rightarrow (a + \alpha t, b + \beta t)$ . If we set  $\varepsilon = \alpha t$  and  $\delta = \beta t$  in  $\textcircled{1}$ , we get



$$\begin{aligned} \text{Relative error} &= \frac{f(a + \alpha t, b + \beta t) - f(a, b) - (p\alpha t + q\beta t)}{\sqrt{\alpha^2 t^2 + \beta^2 t^2}} \\ &= \frac{f(a + \alpha t, b + \beta t) - f(a, b)}{t\sqrt{\alpha^2 + \beta^2}} - p\alpha - q\beta \\ &= \frac{f(a + \alpha t, b + \beta t) - f(a, b)}{t} - p\alpha - q\beta \end{aligned}$$

$$\textcircled{4} \quad \text{Since } \sqrt{\alpha^2 + \beta^2} = 1$$

Assuming  $p = f_x(a, b)$  and  $q = f_y(a, b)$ , we modify  $\textcircled{4}$  as follows:

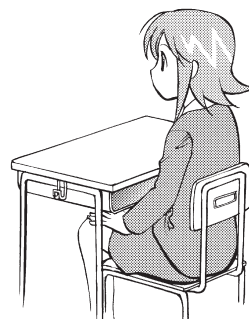
$$\textcircled{5} \quad \frac{f(a + \alpha t, b + \beta t) - f(a, b + \beta t)}{t} + \frac{f(a, b + \beta t) - f(a, b)}{t} - f_x(a, b)\alpha - f_y(a, b)\beta$$

Since the derivative of  $f(x, b + \beta t)$ , a function of  $x$  only, at  $x = a$  is

$$f_x(a, b + \beta t)$$

we get, from the imitating single-variable linear function,

$$f(a + \alpha t, b + \beta t) - f(a, b + \beta t) \approx f_x(a, b + \beta t)\alpha t$$



\* We have calculated the imitating linear function in such a way that its relative error approaches 0 when  $AP \rightarrow 0$  in the  $x$  or  $y$  direction. It is not apparent, however, if the relative error  $\rightarrow 0$  when  $AP \rightarrow 0$  in any direction for the linear function that is made up of the derivatives  $f_x(a, b)$  and  $f_y(a, b)$ . We'll now look into this in detail, although the discussion here will not be so strict.

Similarly, for  $y$  we get

$$f(a, b + \beta t) - f(a, b) \approx f_y(a, b) \beta t$$

Substituting this in ⑤,

$$\begin{aligned} \textcircled{5} &\approx f_x(a, b + \beta t)\alpha + f_y(a, b)\beta - f_x(a, b)\alpha - f_y(a, b)\beta \\ &= (f_x(a, b + \beta t) - f_x(a, b))\alpha \end{aligned}$$

Since  $f_x(a, b + \beta t) - f_x(a, b) \approx 0$  if  $t$  is close enough to 0, the relative error =  $\textcircled{5} \approx 0$ . Thus, we have shown “the relative error  $\rightarrow 0$  when  $AP \rightarrow 0$  in any direction.”

It should be noted that  $f_x$  must be continuous to say  $f_x(a, b + \beta t) - f_x(a, b) \approx 0$  ( $t \approx 0$ ). Unless it is continuous, we don't know whether the derivative exists in every direction, even though  $f_x$  and  $f_y$  exist. Since such functions are rather exceptional, however, we won't cover them in this book.

EXAMPLES (FUNCTION OF EXAMPLE 1 FROM PAGE 183)

Let's find the partial derivatives of  $h(v, t) = vt - 4.9t^2$  at  $(v, t) = (100, 5)$ .

In the  $v$  direction, we differentiate  $h(v, 5) = 5v - 122.5$  and get

$$\frac{\partial h}{\partial v}(v, 5) = 5$$

Thus,

$$\frac{\partial h}{\partial v}(100, 5) = h_v(100, 5) = 5$$



In the  $t$  direction, we differentiate  $h(100, t) = 100t - 4.9t^2$  and get

$$\frac{\partial h}{\partial t}(100, t) = 100 - 9.8t$$

$$\frac{\partial h}{\partial t}(100, 5) = h_t(100, 5) = 100 - 9.8 \times 5 = 51$$

And the imitating linear function is

$$L(x, y) = 5(v - 100) + 51(t - 5) - 377.5$$

In general,

$$\frac{\partial h}{\partial v} = t, \frac{\partial h}{\partial v} = v - 9.8t$$

Therefore, from **Ⓔ** on page 194, near  $(v, t) = (v_0, t_0)$ ,

$$h(v, t) \approx t_0(v - v_0) + (v_0 - 9.8t_0)(t - t_0) + h(v_0, t_0)$$

Next, we'll try imitating the concentration of sugar syrup given  $y$  grams of sugar in  $x$  grams of water.

$$f(x, y) = \frac{100y}{x + y}$$

$$\frac{\partial f}{\partial x} = f_x = -\frac{100y}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = f_y = \frac{100(x + y) - 100y \times 1}{(x + y)^2} = \frac{100x}{(x + y)^2}$$

Thus, near  $(x, y) = (a, b)$ , we have

$$f(x, y) \approx -\frac{100b}{(a + b)^2}(x - a) + \frac{100a}{(a + b)^2}(y - b) + \frac{100b}{a + b}$$

## DEFINITION OF PARTIAL DIFFERENTIATION

When  $z = f(x, y)$  is partially differentiable with respect to  $x$  for every point  $(x, y)$  in a region, the function  $(x, y) \rightarrow f_x(x, y)$ , which relates  $(x, y)$  to  $f_x(x, y)$ , the partial derivative at that point with respect to  $x$ , is called the partial differential function of  $z = f(x, y)$  with respect to  $x$  and can be expressed by any of the following:

$$f_x, f_x(x, y), \frac{\partial f}{\partial x}, \frac{\partial z}{\partial x}$$

Similarly, when  $z = f(x, y)$  is partially differentiable with respect to  $y$  for every point  $(x, y)$  in the region, the function

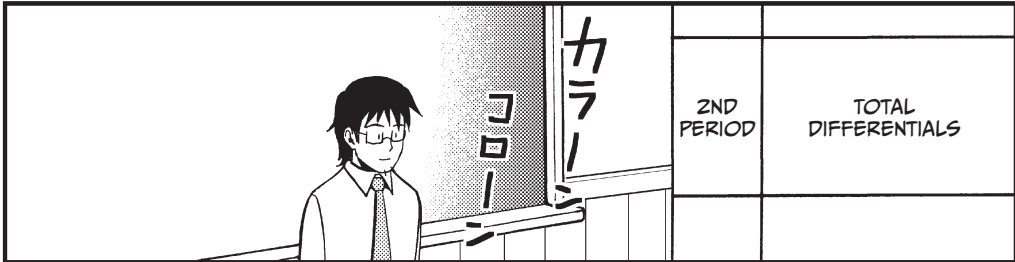
$$(x, y) \rightarrow f_y(x, y)$$

is called the partial differential function of  $z = f(x, y)$  with respect to  $y$  and is expressed by any of the following:

$$f_y, f_y(x, y), \frac{\partial f}{\partial y}, \frac{\partial z}{\partial y}$$

Obtaining the partial derivatives of a function is called *partially differentiating* it.

## TOTAL DIFFERENTIALS



From the imitating linear function of  $z = f(x, y)$  at  $(x, y) = (a, b)$ , we have found

$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

We now modify this as

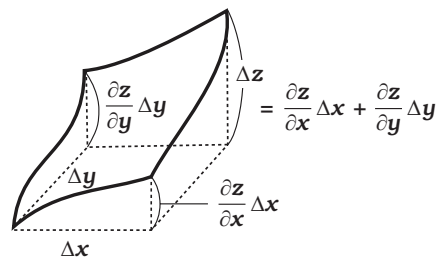
$$\textcircled{6} \quad f(x, y) - f(a, b) \approx \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Since  $f(x, y) - f(a, b)$  means the increment of  $z = f(x, y)$  when a point moves from  $(a, b)$  to  $(x, y)$ , we write this as  $\Delta z$ , as we did for the single-variable functions.

Also,  $(x - a)$  is  $\Delta x$  and  $(y - b)$  is  $\Delta y$ .

Then, expression  $\textcircled{6}$  can be written as

$$\textcircled{7} \quad \Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$$



This expression means, "If  $x$  increases from  $a$  by  $\Delta x$  and  $y$  from  $b$  by  $\Delta y$  in  $z = f(x, y)$ ,  $z$  increases by

$$\frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$$

Since  $\frac{\partial z}{\partial x} \Delta x$  is “the increment of  $z$  in the  $x$  direction when  $y$  is fixed at  $b$ ” and  $\frac{\partial z}{\partial y} \Delta y$  is “the increment in the  $y$  direction when  $x$  is fixed at  $a$ ,” expression ⑦ also means “the increment of  $z = f(x, y)$  is the sum of the increment in the  $x$  direction and that in the  $y$  direction.”

When expression ⑦ is idealized (made instantaneous), we have

$$\textcircled{8} \quad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or

$$\textcircled{9} \quad df = f_x dx + f_y dy$$

EXPRESSION ⑧ OR ⑨ IS CALLED THE FORMULA OF THE TOTAL DIFFERENTIAL.



( $\Delta$  has been changed to  $d$ .)

The meaning of the formula is as follows.

Increment of height of a curved surface =

Partial derivative in the  $x$  direction  $\times$  Increment in the  $x$  direction  $+$  Partial derivative in the  $y$  direction  $\times$  Increment in the  $y$  direction

Now, let’s look at the expression of a total differential from Example 4 (page 183).

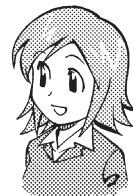
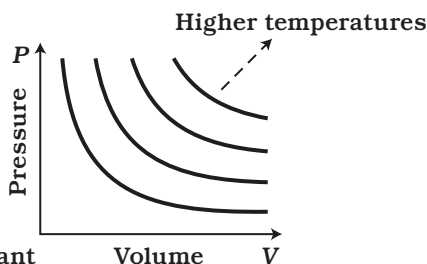
By converting the unit properly, we rewrite the equation of temperature as  $T = PV$ .

$$\frac{\partial T}{\partial P} = \frac{\partial(PV)}{\partial P} = V \quad \text{and} \quad \frac{\partial T}{\partial V} = \frac{\partial(PV)}{\partial V} = P$$

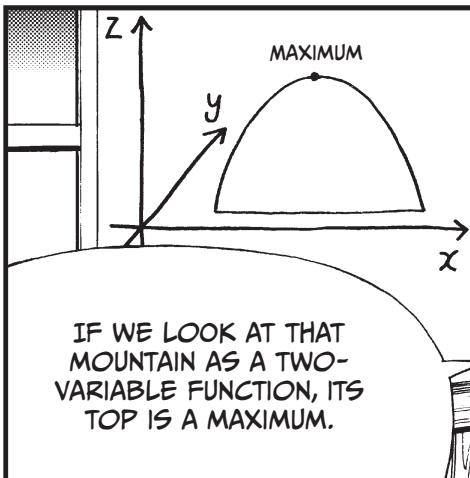
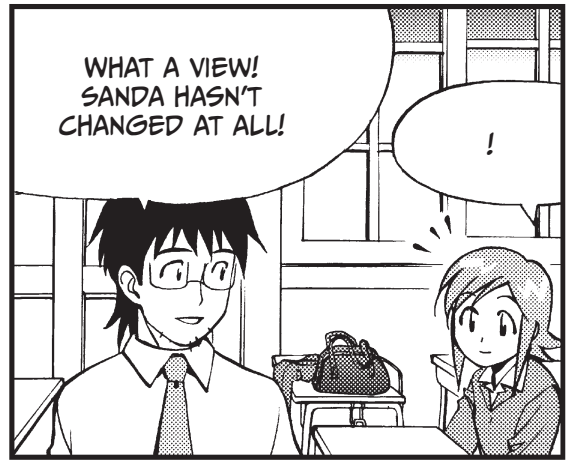
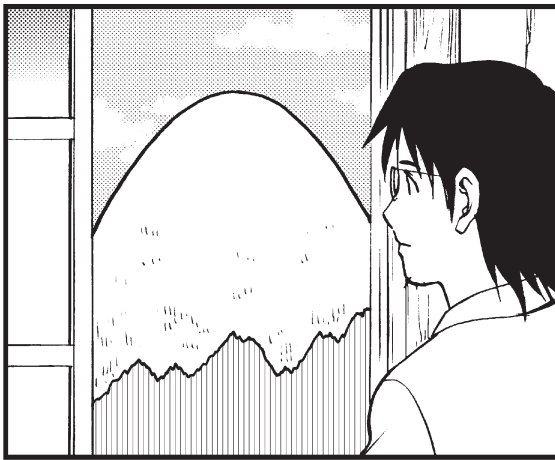
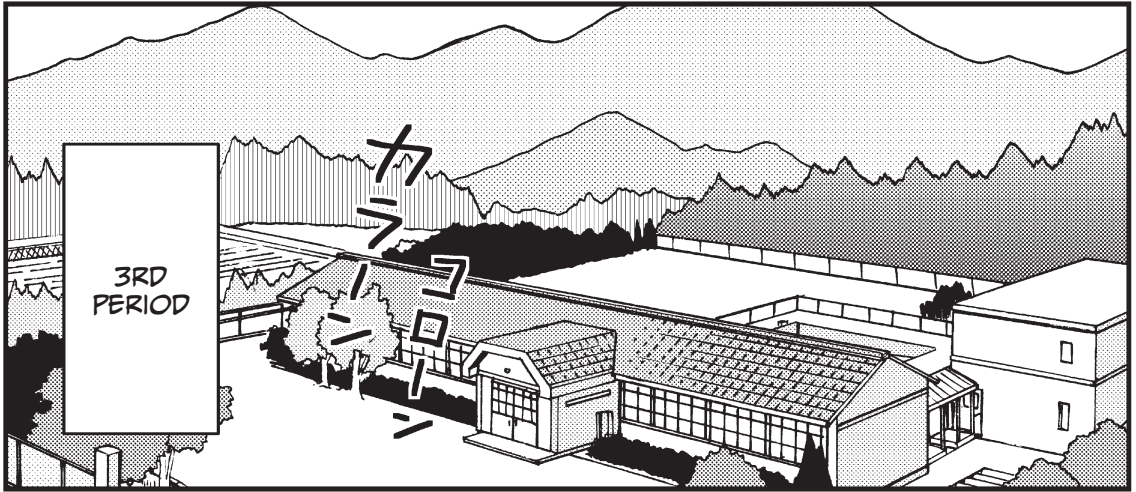
Thus, the total differential can be written as  $dT = VdP + PdV$ .

In the form of an approximate expression, this is  $\Delta T \approx V\Delta P + P\Delta V$ .

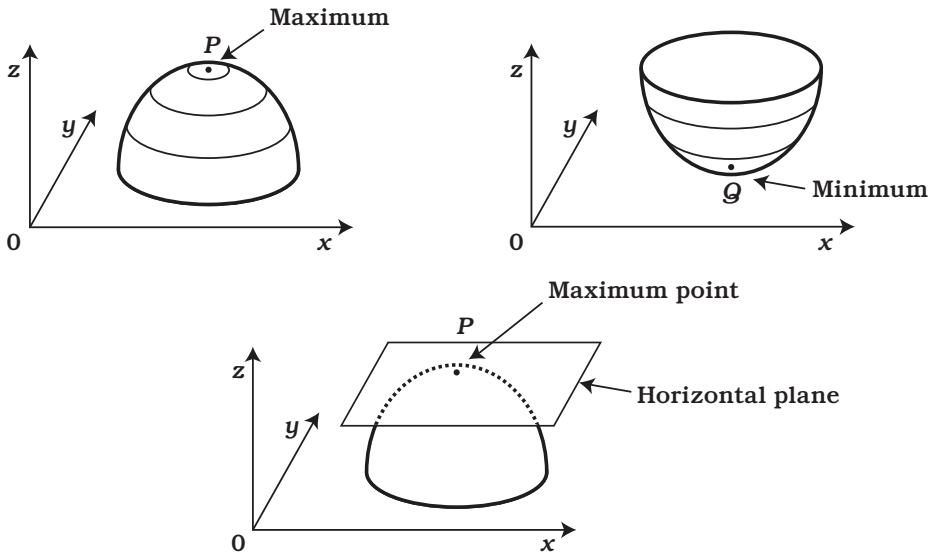
THIS MEANS THAT FOR AN IDEAL GAS, THE INCREMENT OF TEMPERATURE CAN BE CALCULATED BY THE VOLUME TIMES THE INCREMENT OF PRESSURE PLUS THE PRESSURE TIMES THE INCREMENT OF VOLUME.



# CONDITIONS FOR EXTREMA



The *extrema* of a two-variable function  $f(x, y)$  are where its graph is at the top of a mountain or the bottom of a valley.



Since the plane tangent to the graph at point  $P$  or  $Q$  is parallel to the  $x$ - $y$  plane, we should have

$$f(x, y) \approx p(x - a) + q(y - b) + f(a, b)$$

with  $p = q = 0$  in the imitating linear function.

Since

$$p = \frac{\partial f}{\partial x} (= f_x) \quad q = \frac{\partial f}{\partial y} (= f_y)$$

the condition for extrema\* is, if  $f(x, y)$  has an extremum at  $(x, y) = (a, b)$ ,

$$f_x(a, b) = f_y(a, b) = 0$$

or

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$$

---

\* The opposite of this is not true. In other words, even if  $f_x(a, b) = f_y(a, b) = 0$ ,  $f$  will not always have an extremum at  $(x, y) = (a, b)$ . Thus, this condition only picks up the candidates for extrema.



AT THE EXTREMA OF A TWO-VARIABLE FUNCTION, THE PARTIAL DERIVATIVES IN BOTH THE  $x$  AND  $y$  DIRECTIONS ARE ZERO.

EXAMPLE

Let's find the minimum of  $f(x, y) = (x - y)^2 + (y - 2)^2$ . First, we'll find it algebraically.

Since

$$(x - y)^2 \geq 0 \quad (y - 2)^2 \geq 0$$

$$f(x, y) = (x - y)^2 + (y - 2)^2 \geq 0$$

If we substitute  $x = y = 2$  here,

$$f(2, 2) = (2 - 2)^2 + (2 - 2)^2 = 0$$

From this,  $f(x, y) \geq f(2, 2)$  for all  $(x, y)$ . In other words,  $f(x, y)$  has a minimum of zero at  $(x, y) = (2, 2)$ .

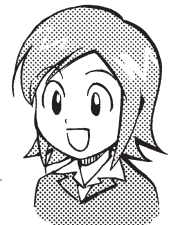
On the other hand,  $\frac{\partial f}{\partial x} = 2(x - y)$  and  $\frac{\partial f}{\partial y} = 2(x - y)(-1) + 2(y - 2) = -2x + 4y - 4$ . If we set

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

and solve these simultaneous equations,

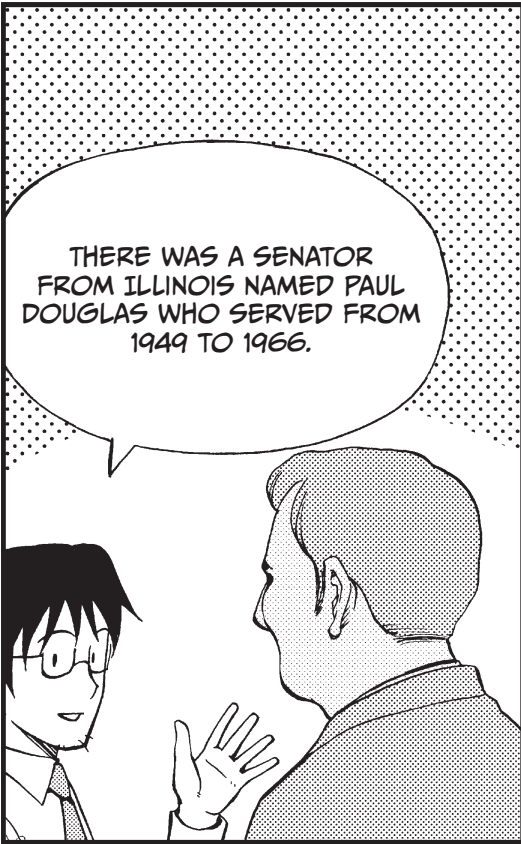
$$\begin{cases} 2x - 2y = 0 \\ -2x + 4y - 4 = 0 \end{cases}$$

we find that  $(x, y) = (2, 2)$ , just as we found above.

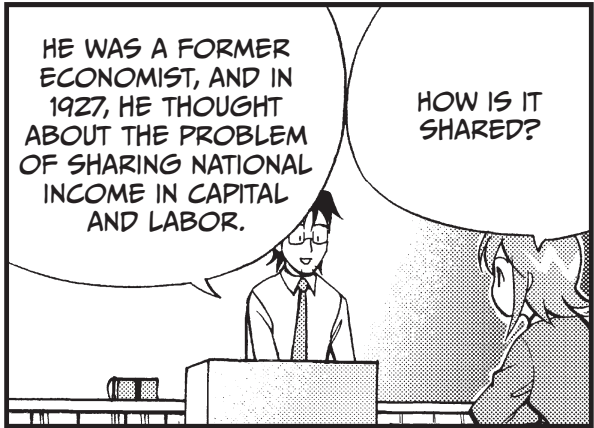


THE SOLUTIONS ARE THE SAME!

# APPLYING PARTIAL DIFFERENTIATION TO ECONOMICS

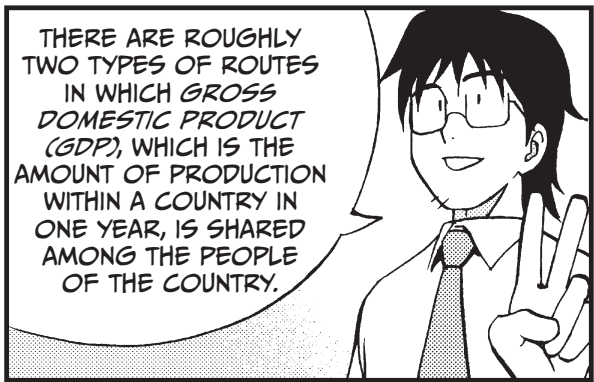


THERE WAS A SENATOR FROM ILLINOIS NAMED PAUL DOUGLAS WHO SERVED FROM 1949 TO 1966.



HE WAS A FORMER ECONOMIST, AND IN 1927, HE THOUGHT ABOUT THE PROBLEM OF SHARING NATIONAL INCOME IN CAPITAL AND LABOR.

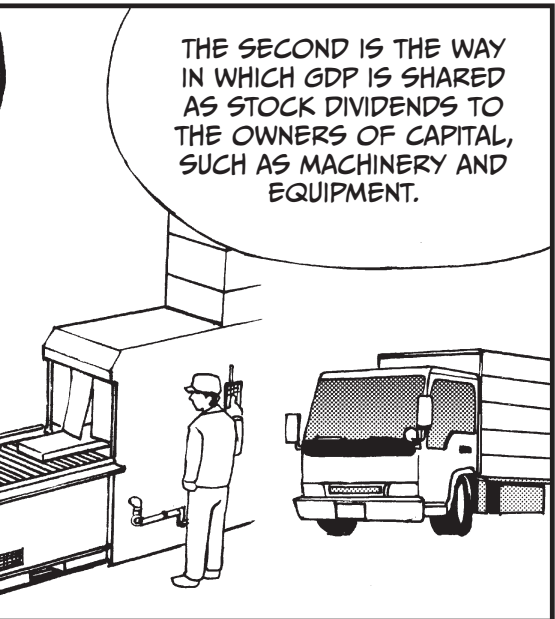
HOW IS IT SHARED?



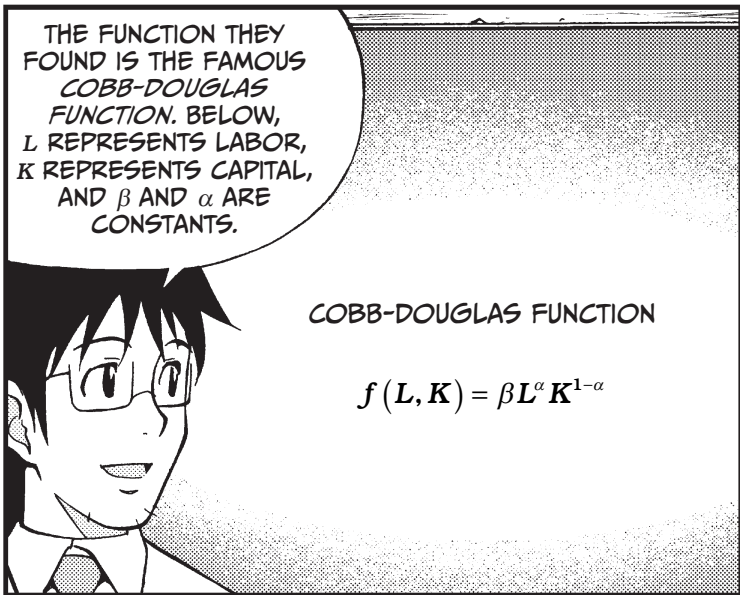
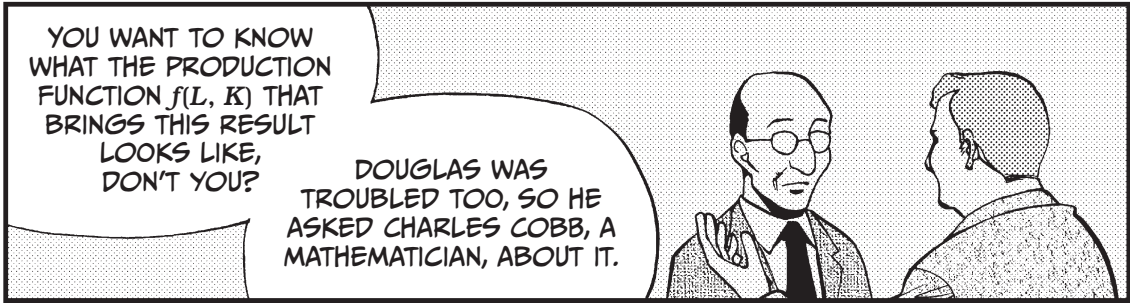
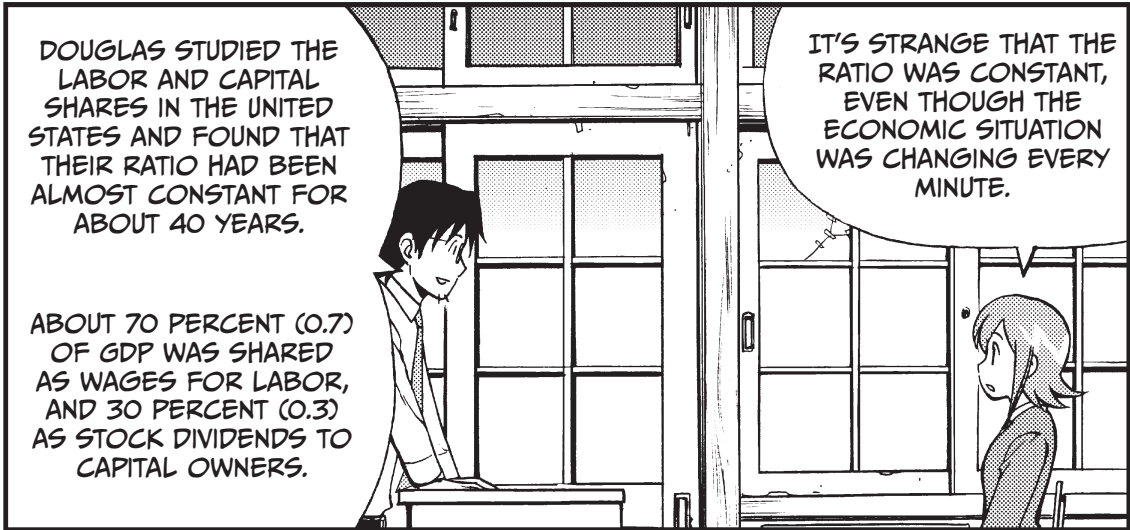
THERE ARE ROUGHLY TWO TYPES OF ROUTES IN WHICH *GROSS DOMESTIC PRODUCT (GDP)*, WHICH IS THE AMOUNT OF PRODUCTION WITHIN A COUNTRY IN ONE YEAR, IS SHARED AMONG THE PEOPLE OF THE COUNTRY.



THE FIRST ONE IS THE WAY IN WHICH GDP IS SHARED AS WAGES FOR LABOR.



THE SECOND IS THE WAY IN WHICH GDP IS SHARED AS STOCK DIVIDENDS TO THE OWNERS OF CAPITAL, SUCH AS MACHINERY AND EQUIPMENT.



First, let's suppose that wages are measured in units of  $w$ , and capital is measured in units of  $r$ . We'll consider the production of the entire country to be given by the function  $f(L, K)$  and assume the country is acting as a profit-maximizing business. The profit  $P$  is given by the equation:

$$P = f(L, K) - wL - rK$$

Because we know that a business chooses values of  $L$  and  $K$  to maximize profit ( $P$ ), we get the following condition for extrema:

$$\frac{\partial P}{\partial L} = \frac{\partial P}{\partial K} = 0$$

$$\textcircled{1} \quad 0 = \frac{\partial P}{\partial L} = \frac{\partial f}{\partial L} - \frac{\partial(wL)}{\partial L} - \frac{\partial(rK)}{\partial L} = \frac{\partial f}{\partial L} - w \Rightarrow w = \frac{\partial f}{\partial L}$$

$$\textcircled{2} \quad 0 = \frac{\partial P}{\partial K} = \frac{\partial f}{\partial K} - \frac{\partial(wL)}{\partial K} - \frac{\partial(rK)}{\partial K} = \frac{\partial f}{\partial K} - r \Rightarrow r = \frac{\partial f}{\partial K}$$

The relations far to the right mean the following.

Wages = Partial derivative of the production function with respect to  $L$

Capital share = Partial derivative of the production function with respect to  $K$

Now, the reward the people of the country receive for labor is Wage  $\times$  Labor =  $wL$ . When this is 70 percent of GDP, we have

$$\textcircled{3} \quad wL = 0.7 f(L, K)$$

Similarly, the reward the capital owners receive is

$$\textcircled{4} \quad rK = 0.3 f(L, K)$$

From  $\textcircled{1}$  and  $\textcircled{3}$ ,

$$\textcircled{5} \quad \frac{\partial f}{\partial L} \times L = 0.7 f(L, K)$$

From  $\textcircled{2}$  and  $\textcircled{4}$ ,

$$\textcircled{6} \quad \frac{\partial f}{\partial K} \times K = 0.3 f(L, K)$$



Cobb found  $f(L, K)$  that satisfies these equations. It is

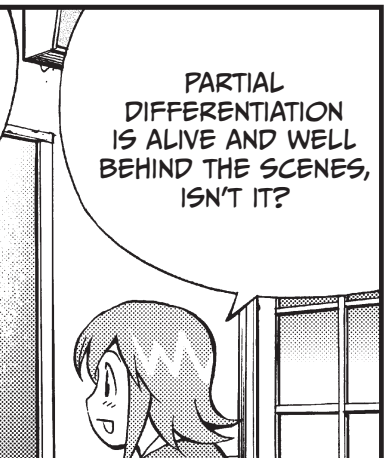
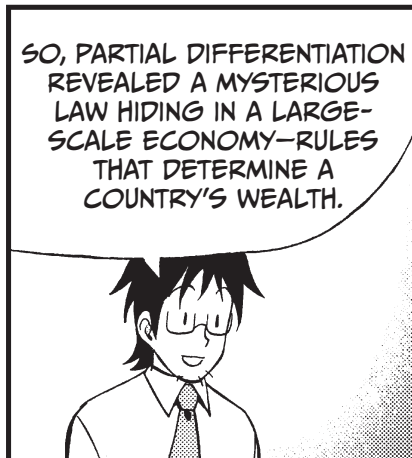
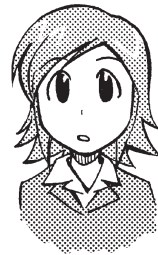
$$f(L, K) = \beta L^{0.7} K^{0.3}$$

where  $\beta$  is a positive parameter meaning the level of technology.

Let's check if this satisfies the above conditions.

$$\begin{aligned}\frac{\partial f}{\partial L} \times L &= \frac{\partial(\beta L^{0.7} K^{0.3})}{\partial L} \times L = 0.7\beta L^{(-0.3)} K^{0.3} \times L^1 \\ &= 0.7\beta L^{0.7} K^{0.3} \\ &= 0.7f(L, K)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial K} \times K &= \frac{\partial(\beta L^{0.7} K^{0.3})}{\partial K} \times K = 0.3\beta L^{0.7} K^{(-0.7)} \times K^1 \\ &= 0.3\beta L^{0.7} K^{0.3} \\ &= 0.3f(L, K)\end{aligned}$$



## THE CHAIN RULE

We have seen single-variable composite functions before (page 14).

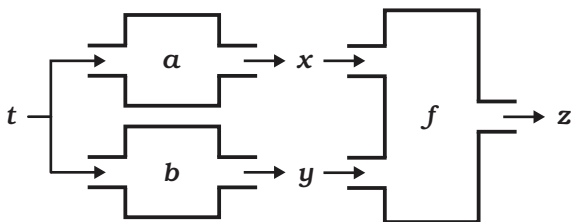
$$y = f(x), z = g(y), z = g(f(x)),$$

$$g(f(x))' = g'(f(x))f'(x)$$



HERE, LET'S DERIVE THE FORMULA OF PARTIAL DIFFERENTIATION (THE CHAIN RULE) FOR MULTIVARIABLE COMPOSITE FUNCTIONS.

We assume that  $z$  is a two-variable function of  $x$  and  $y$ , expressed as  $z = f(x, y)$ , and that  $x$  and  $y$  are both single-variable functions of  $t$ , expressed as  $x = a(t)$  and  $y = b(t)$ , respectively. Then,  $z$  can be expressed as a function of  $t$  only, as shown below.



This relationship can be written as

$$z = f(x, y) = f(a(t), b(t))$$

What is the form of  $\frac{dz}{dt}$  then?

We assume  $a(t_0) = x_0$ ,  $b(t_0) = y_0$  and  $f(x_0, y_0) = f(a(t_0), b(t_0)) = z_0$  when  $t = t_0$ , and consider only the vicinities of  $t_0$ ,  $x_0$ ,  $y_0$ , and  $z_0$ .

If we obtain an  $\alpha$  that satisfies

$$\textcircled{1} \quad z - z_0 \approx \alpha \times (t - t_0)$$

it is  $\frac{dz}{dt}(t_0)$ .

First, from the approximation of  $x = a(t)$ ,

$$\textcircled{2} \quad x - x_0 \approx \frac{da}{dt}(t_0)(t - t_0)$$

Similarly, from the approximation of  $y = b(t)$ ,

$$\textcircled{3} \quad y - y_0 \approx \frac{db}{dt}(t_0)(t - t_0)$$

Next, from the formula of total differential for a two-variable function  $f(x, y)$ ,

$$\textcircled{4} \quad z - z_0 \approx \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Substituting  $\textcircled{2}$  and  $\textcircled{3}$  in  $\textcircled{4}$ ,

$$\begin{aligned} \textcircled{5} \quad z - z_0 &\approx \frac{\partial f}{\partial x}(x_0, y_0) \frac{da}{dt}(t_0)(t - t_0) + \frac{\partial f}{\partial y}(x_0, y_0) \frac{db}{dt}(t_0)(t - t_0) \\ &= \left( \frac{\partial f}{\partial x}(x_0, y_0) \frac{da}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0) \frac{db}{dt}(t_0) \right) (t - t_0) \end{aligned}$$

Comparing  $\textcircled{1}$  and  $\textcircled{5}$ , we get

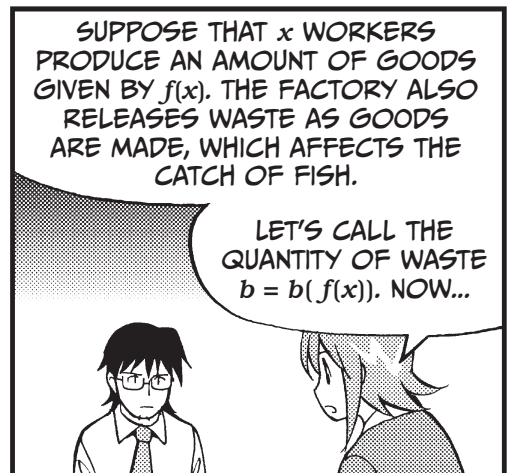
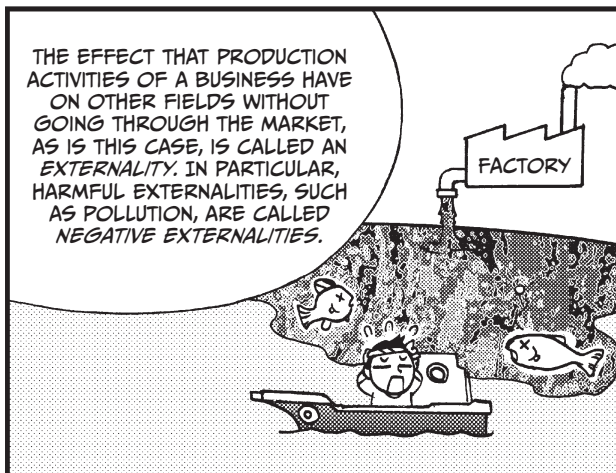
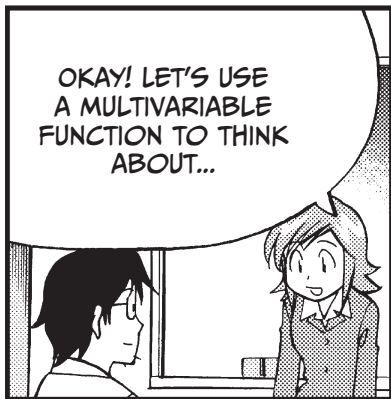
$$\alpha = \frac{\partial f}{\partial x}(x_0, y_0) \frac{da}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0) \frac{db}{dt}(t_0)$$

This is what we wanted, and we now have the following formula!

#### FORMULA 6-1: THE CHAIN RULE

When  $z = f(x, y)$ ,  $x = a(t)$ ,  $y = b(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{da}{dt} + \frac{\partial f}{\partial y} \frac{db}{dt}$$



We assume that the catch of fish can be expressed as a two-variable function  $g(y, b)$  of the amount of labor  $y$  and the amount of waste  $b$ .

(The catch  $g(y, b)$  decreases as  $b$  increases. Thus,  $\frac{\partial g}{\partial b}$  is negative.)

Since the variable  $x$  is contained in  $g(y, b) = g(y, b(f(x)))$ , production at the factory influences fisheries without going through the market. This is an externality.

First, let's see what happens if the factory and the fishery each act (selfishly) only for their own benefit. If the wage is  $w$  for both of them, the price of a commodity produced at the factory  $p$  and the price of a fish  $q$ , the profit for the factory is given by

$$\textcircled{1} \quad P_1(x) = pf(x) - wx$$

Thus, the factory wants to maximize this, and the condition for extrema is

$$\textcircled{2} \quad \frac{dP_1}{dx} = pf'(x) - w = 0 \Leftrightarrow pf'(x) = w$$

Let  $s$  be such  $x$  that satisfies this condition. Thus, we have

$$\textcircled{3} \quad pf'(s) = w$$

This  $s$  is the amount of labor employed by the factory, the amount of production is  $f(s)$ , and the amount of waste is given by

$$b^* = b(f(s))$$

Next, the profit  $P_2$  for the fishery is given by

$$P_2 = qg(y, b) - wy$$

Since the amount of waste from the factory is given by  $b^* = b(f(s))$ ,

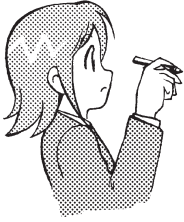
$$\textcircled{4} \quad P_2 = qg(y, b^*) - wy$$

which is practically a single-variable function of  $y$ . To maximize  $P_2$ , we use only the condition about  $y$  for extrema of a two-variable function.

$$\textcircled{5} \quad \frac{\partial P_2}{\partial y} = q \frac{\partial g}{\partial y}(y, b^*) - w = 0 \Leftrightarrow q \frac{\partial g}{\partial y}(y, b^*) = w$$

Therefore, the optimum amount of labor  $t$  to be input satisfies

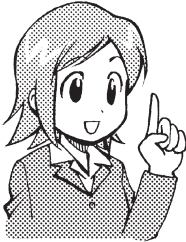
$$\textcircled{6} \quad q \frac{\partial g}{\partial y}(t, b^*) = w$$



IN SUMMARY...

The production at the factory and the catch in the fishery when they act freely in this model are given by  $f(s)$  and  $g(t, b^*)$ , respectively, where  $s$  and  $t$  satisfy the following.

$$\begin{aligned} \textcircled{3} \quad & pf'(s) = w \\ \textcircled{6} \quad & b^* = b(f(s)), q \frac{\partial g}{\partial y}(t, b^*) = w \end{aligned}$$



NOW, MR. SEKI, LET'S CHECK IF THIS IS THE BEST RESULT FOR THE WHOLE SOCIETY. IF WE TAKE BOTH THE FACTORY AND THE FISHERY INTO ACCOUNT, WE SHOULD MAXIMIZE THE SUM OF THE PROFIT FOR BOTH.

$$P_3 = pf(x) + qg(y, b(f(x))) - wx - wy$$

Since  $P_3$  is a two-variable function of  $x$  and  $y$ , the condition for extrema is given by

$$\frac{\partial P_3}{\partial x} = \frac{\partial P_3}{\partial y} = 0$$

The first partial derivative is obtained as follows.

$$\begin{aligned} \frac{\partial P_3}{\partial x} &= pf'(x) + q \frac{\partial g(y, b(f(x)))}{\partial x} - w \\ &= pf'(x) + q \frac{\partial g(y, b(f(x)))}{\partial b} b'(f(x)) f'(x) - w \end{aligned}$$

(Here, we used the chain rule.)

Thus,

$$\frac{\partial P_3}{\partial x} = 0 \Leftrightarrow \left( p + q \frac{\partial g}{\partial b}(y, b(f(x))) b'(f(x)) \right) f'(x) = w$$

Similarly,

$$\textcircled{8} \quad \frac{\partial P_3}{\partial y} = 0 \Leftrightarrow q \frac{\partial g}{\partial y}(y, b(f(x))) = w$$

Thus, if the optimum amount of labor is  $S$  for the factory and  $T$  for the fishery, they satisfy

$$\textcircled{9} \quad \left( p + q \frac{\partial g}{\partial b}(T, b(f(S))) b'(f(S)) \right) f'(S) = w$$

$$\textcircled{10} \quad q \frac{\partial g}{\partial y}(T, b(f(S))) = w$$

Although these equations look complicated, they are really just two-variable simultaneous equations.

If we compare these equations with equations  $\textcircled{3}$  and  $\textcircled{6}$ , we find that  $\textcircled{3}$  and  $\textcircled{9}$  are different while  $\textcircled{6}$  and  $\textcircled{10}$  are the same. Then, how do they differ?

$$\textcircled{3} \quad p \times f'(s) = w$$

$$\textcircled{11} \quad (p + \heartsuit) \times f'(S) = w$$

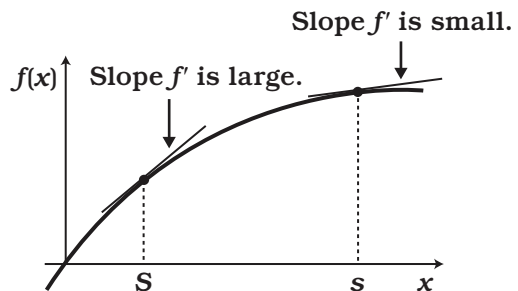
As you see here,  $\heartsuit$  has appeared in the expression.

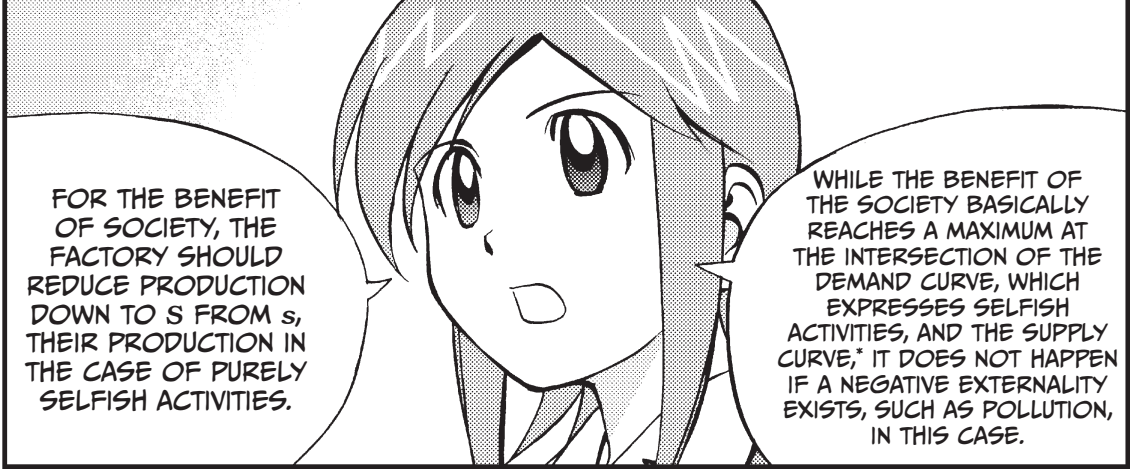
Since  $\left( \heartsuit = q \frac{\partial g}{\partial b} b'(f(S)) \right)$  is negative,  $p + \heartsuit$  is smaller than  $p$ .

Since  $f'(S)$  or  $f'(s)$  is multiplied to the first part to give the same value  $w$ ,  $f'(S)$  must be larger than  $f'(s)$ .

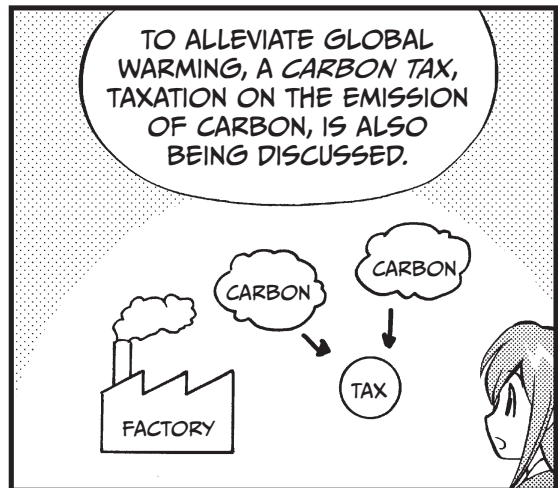
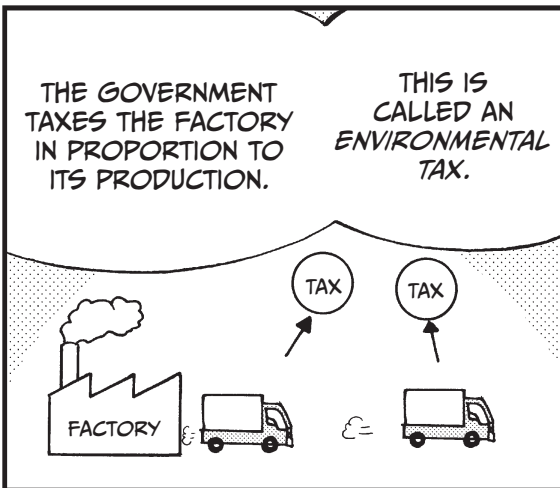
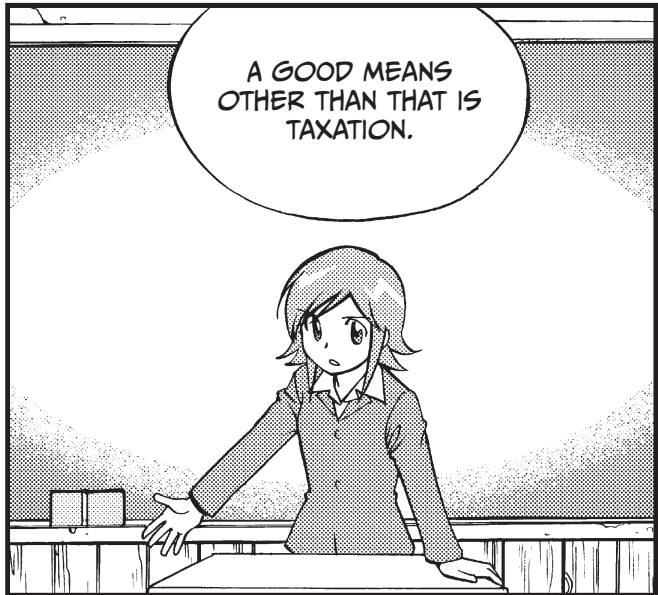


NOW, SINCE THE GRAPH OF  $f(x)$  GENERALLY LOOKS LIKE THIS,





\* SEE PAGE 105.



LET'S ASSUME THAT THE TAX ON A UNIT COMMODITY PRODUCED AT THE FACTORY IS

-♥.

$$-\heartsuit = -q \frac{\partial g}{\partial b} b'(f(s))$$

THIS IS A POSITIVE CONSTANT.



THEN, THE PROFIT ① IN THE CASE OF SELFISH ACTIVITIES BECOMES LIKE THIS.

$$\textcircled{1} P_1(x) = pf(x) - wx - (-\heartsuit f(x))$$

THE CONDITION FOR EXTREMA THAT MAXIMIZE THIS IS...

$$\textcircled{2} \frac{\partial P_1}{\partial x} = pf'(x) - w + \heartsuit f'(x) = 0 \Leftrightarrow (p + \heartsuit) f'(x) = w$$

SINCE ② IS THE SAME EQUATION AS ①, THE PRODUCTION AT THE FACTORY NOW MAXIMIZES THE BENEFIT FOR SOCIETY.

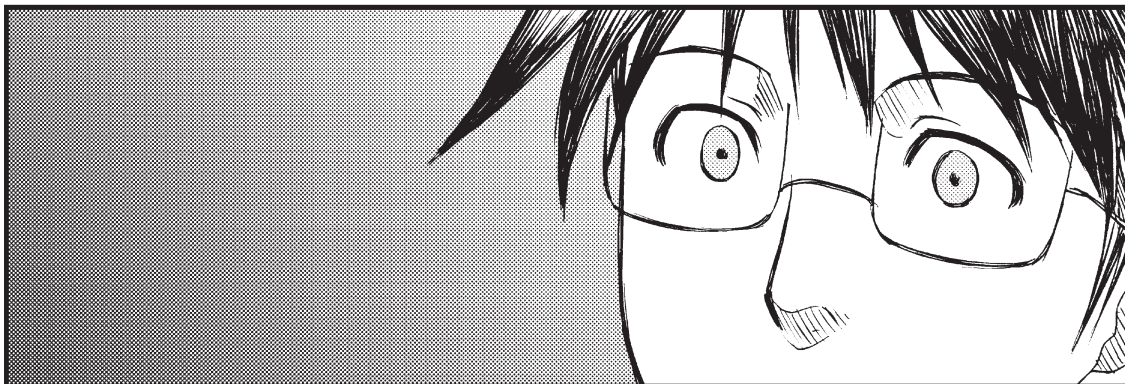
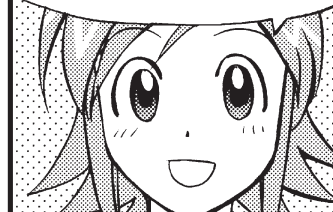


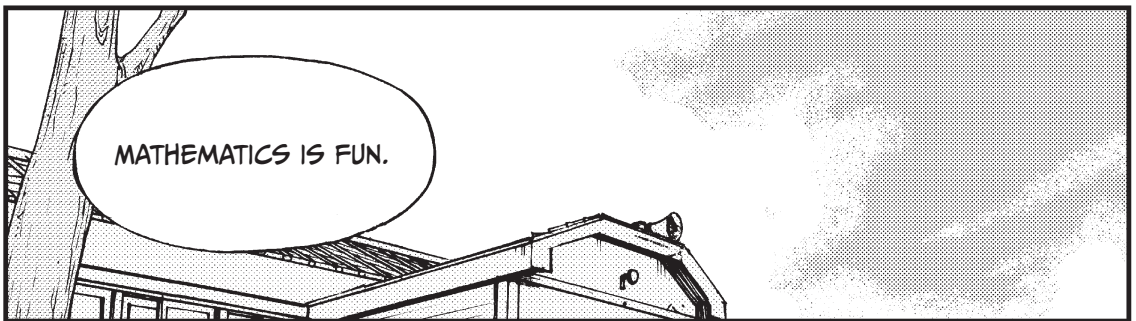
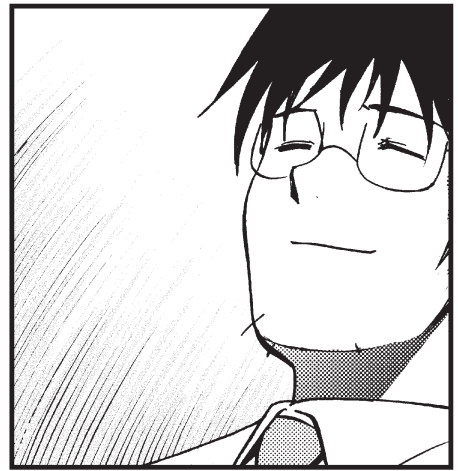
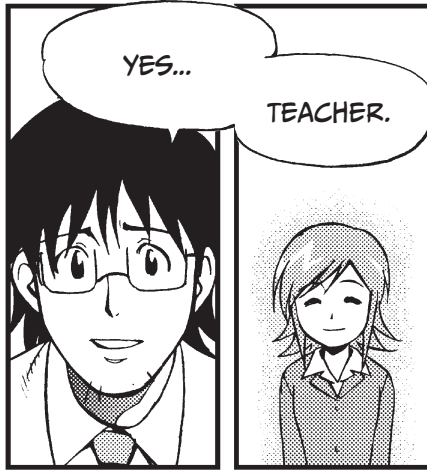
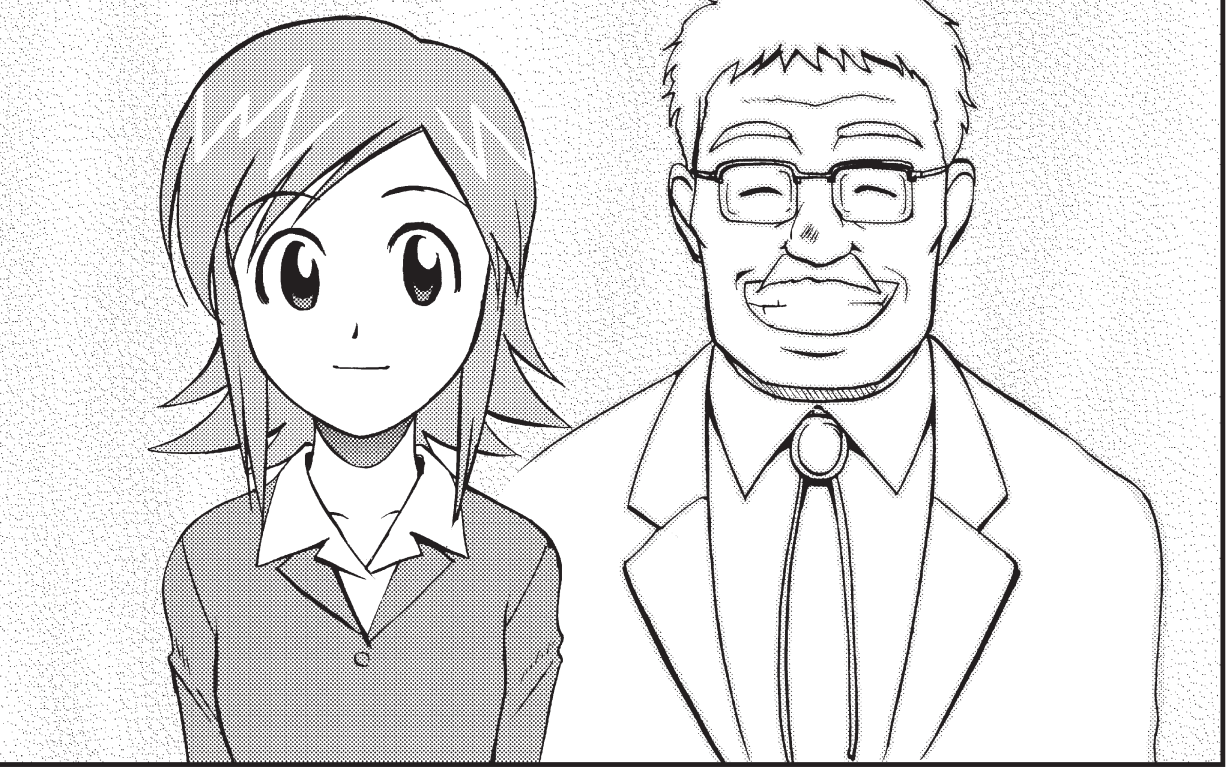
ORDINARY TAXES (INCOME TAX, CONSUMPTION TAX) ARE FOR PUBLIC INVESTMENT...



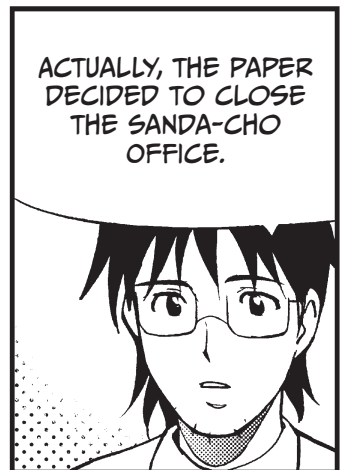
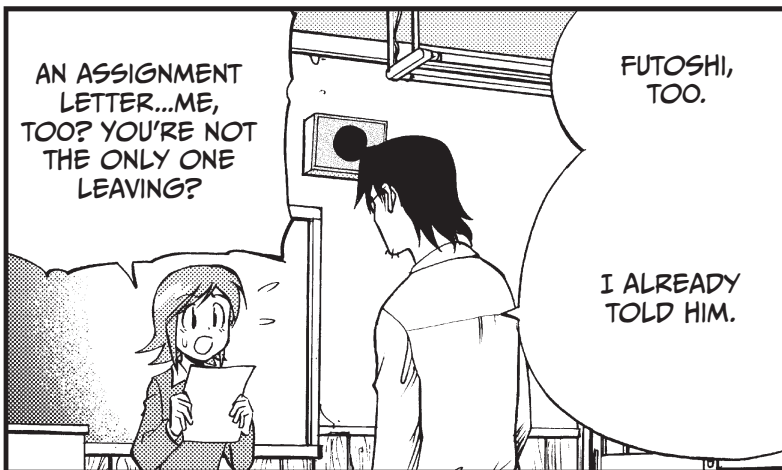
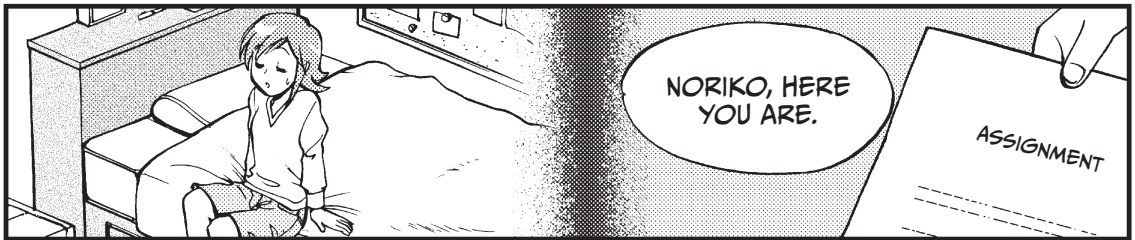
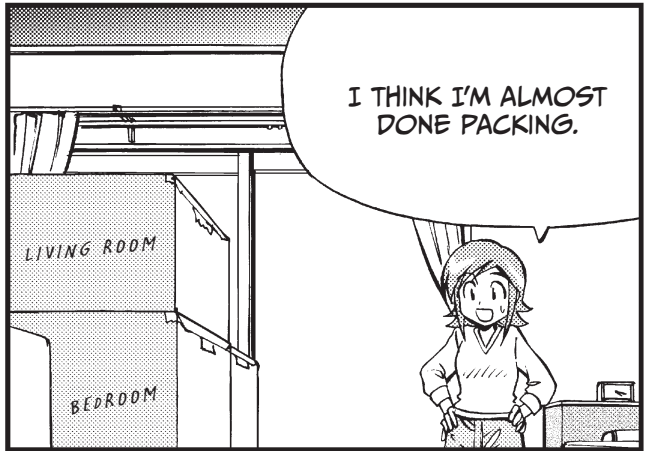
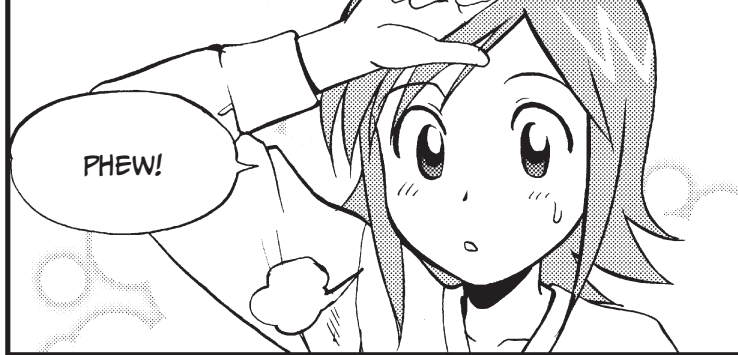
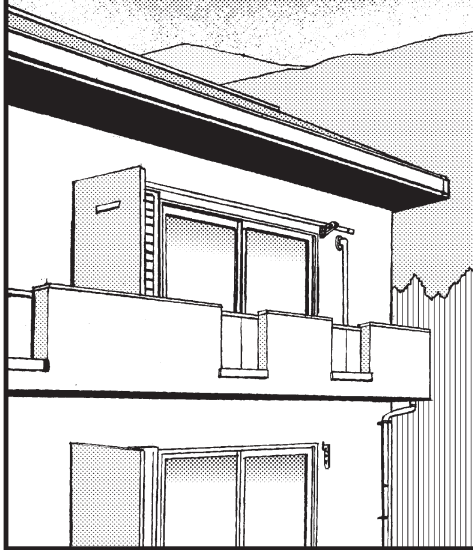
AN ENVIRONMENTAL TAX IS FOR MAINTAINING A HEALTHY ENVIRONMENT BY CONTROLLING THE ECONOMY.

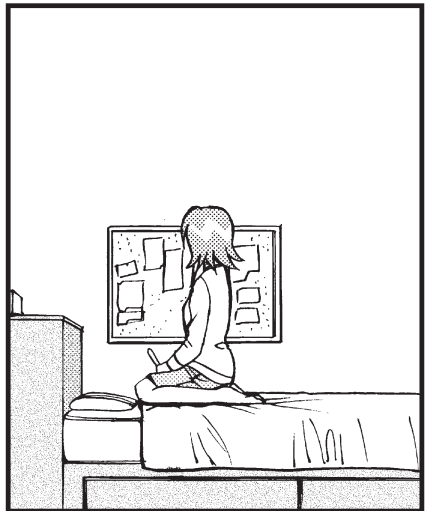
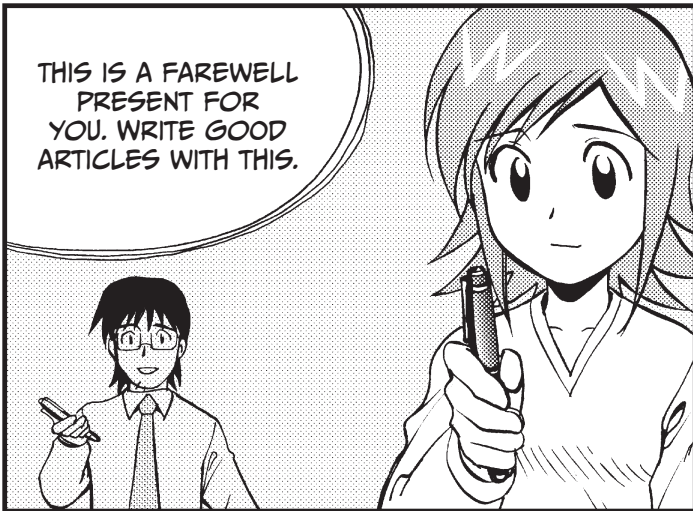
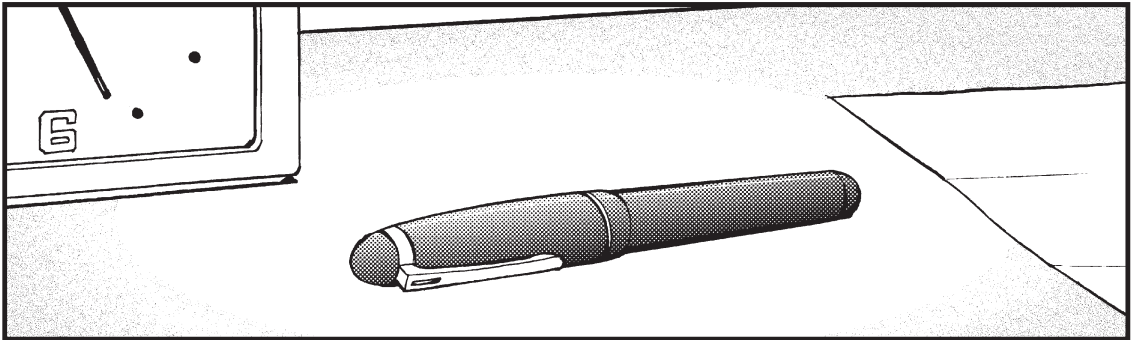
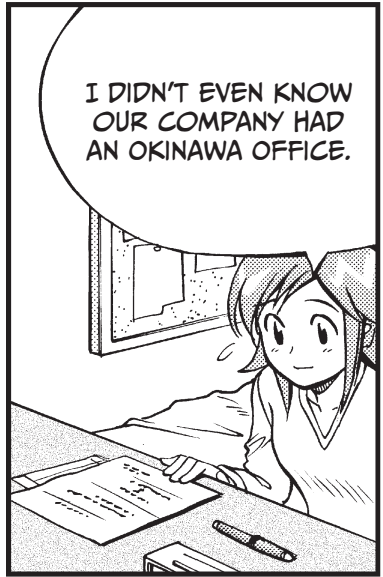
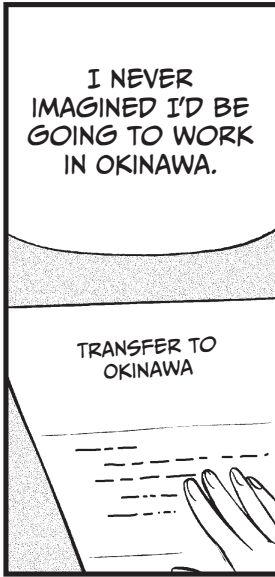
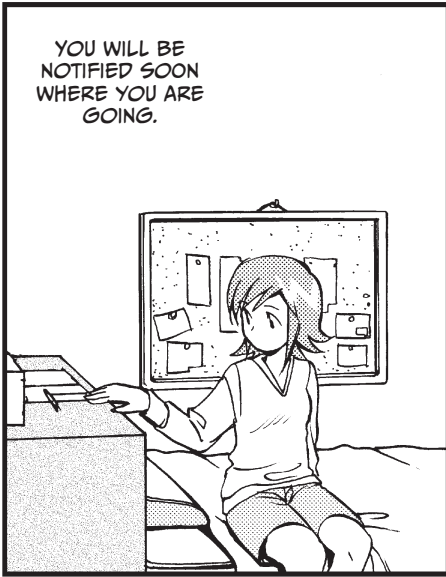
HAVE YOU GOT IT, MR. SEKI?





NORIKO'S APARTMENT  
A FEW DAYS LATER

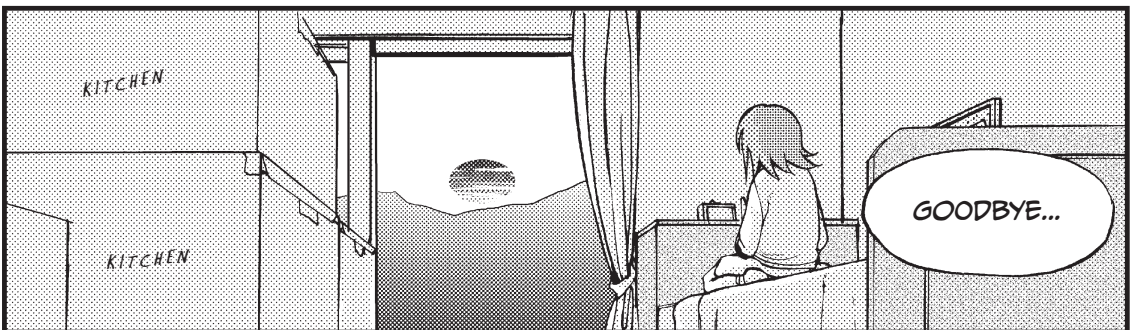
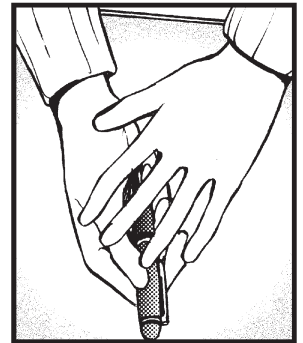
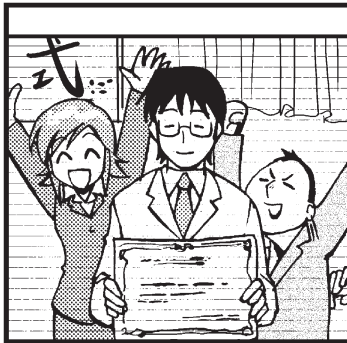
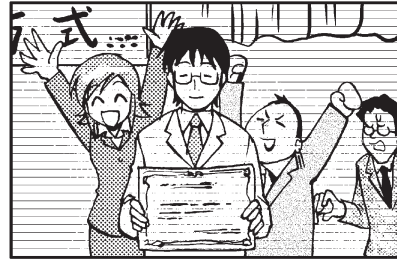




**Burnham  
Chemical  
Apologizes  
Ox Bay Pollution**

**Reconciliation  
with Fishery  
Cooperative  
Expected**

Series: Environment and Economy



## DERIVATIVES OF IMPLICIT FUNCTIONS

A point  $(x, y)$  for which a two-variable function  $f(x, y)$  is equal to constant  $c$  describes a graph given by  $f(x, y) = c$ . When a part of the graph is viewed as a single-variable function  $y = h(x)$ , it is called an *implicit function*. An implicit function  $h(x)$  satisfies  $f(x, h(x)) = c$  for all  $x$  defined. We are going to obtain  $h(x)$  here.

When  $z = f(x, y)$ , the formula of total differentials is written as  $dz = f_x dx + f_y dy$ . If  $(x, y)$  moves on the graph of  $f(x, y) = c$ , the value of the function  $f(x, y)$  does not change, and the increment of  $z$  is 0, that is,  $dz = 0$ . Then, we get  $0 = f_x dx + f_y dy$ . Assuming  $f_y \neq 0$  and modifying this, we get

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

The left side of this equation is the ideal expression of the increment of  $y$  divided by the increment of  $x$  at a point on the graph. It is exactly the derivative of  $h(x)$ . Thus,

$$h'(x) = -\frac{f_x}{f_y}$$

### EXAMPLE

$f(x, y) = r^2$ , where  $f(x, y) = x^2 + y^2$ , describes a circle of radius  $r$  centered at the origin. Near a point that satisfies  $x^2 \neq r^2$ , we can solve  $f(x, y) = x^2 + y^2 = r^2$  to find the implicit function  $y = h(x) = r^2 - x^2$  or  $y = h(x) = -\sqrt{r^2 - x^2}$ . Then, from the formula, the derivative of these functions is given by

$$h'(x) = -\frac{f_x}{f_y} = -\frac{x}{y}$$

## EXERCISES

1. Obtain  $f_x$  and  $f_y$  for  $f(x, y) = x^2 + 2xy + 3y^2$ .
2. Under the gravitational acceleration  $g$ , the period  $T$  of a pendulum having length  $L$  is given by

$$T = 2\pi \sqrt{\frac{L}{g}}$$

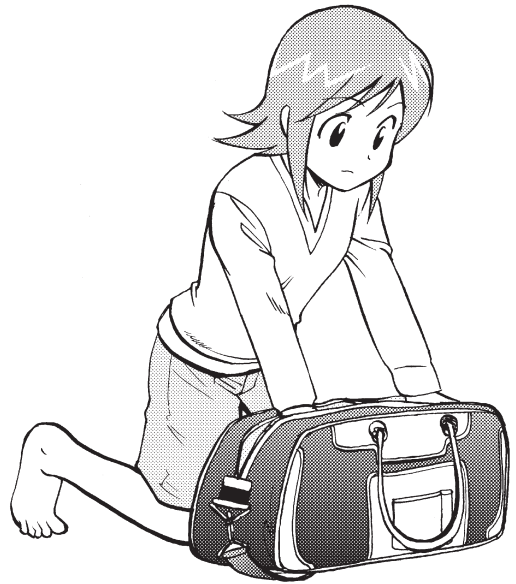
(the gravitational acceleration  $g$  is known to vary depending on the height from the ground).

Obtain the expression for total differential of  $T$ .

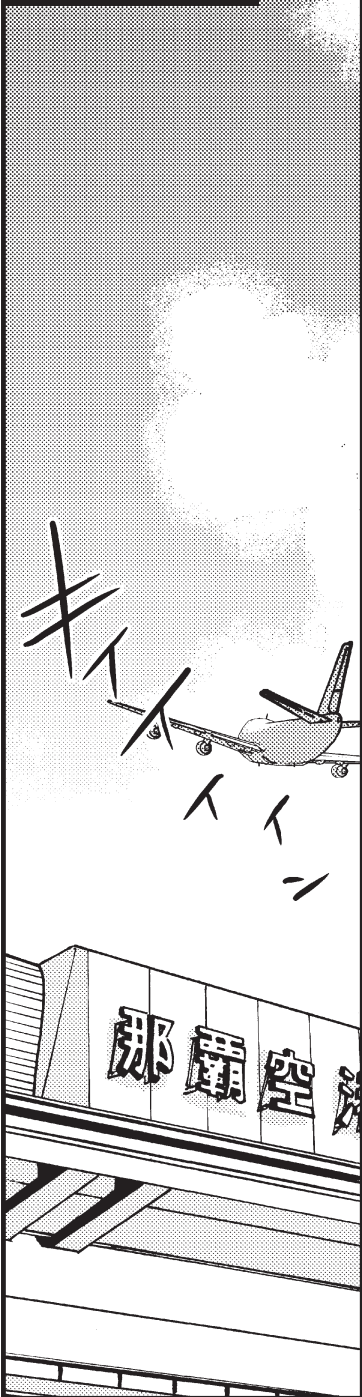
If  $L$  is elongated by 1 percent and  $g$  decreases by 2 percent, about what percentage does  $T$  increase?

3. Using the chain rule, calculate the differential formula of the implicit function  $h(x)$  of  $f(x, y) = c$  in a different way than above.

**EPILOGUE:  
WHAT IS MATHEMATICS FOR?**



NAHA AIRPORT



PHEW, IT'S HOT!

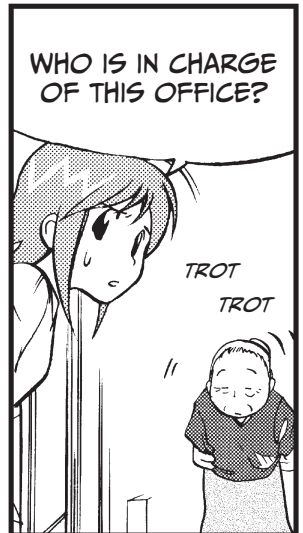
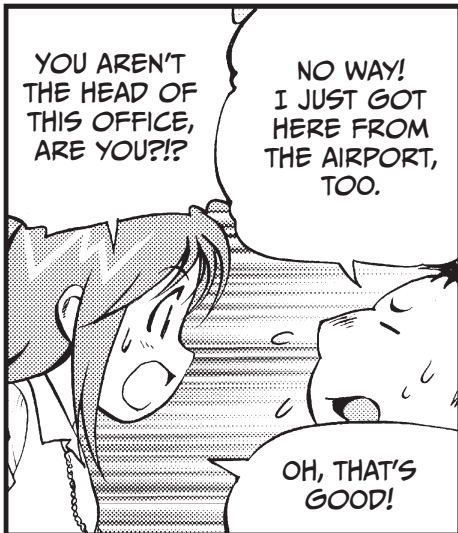
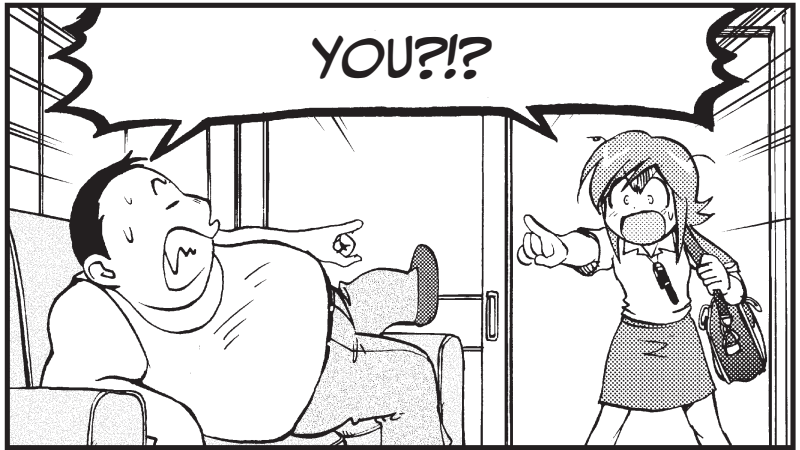
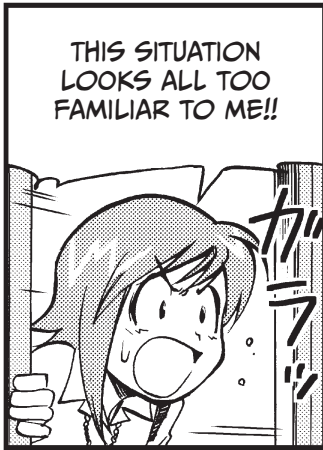
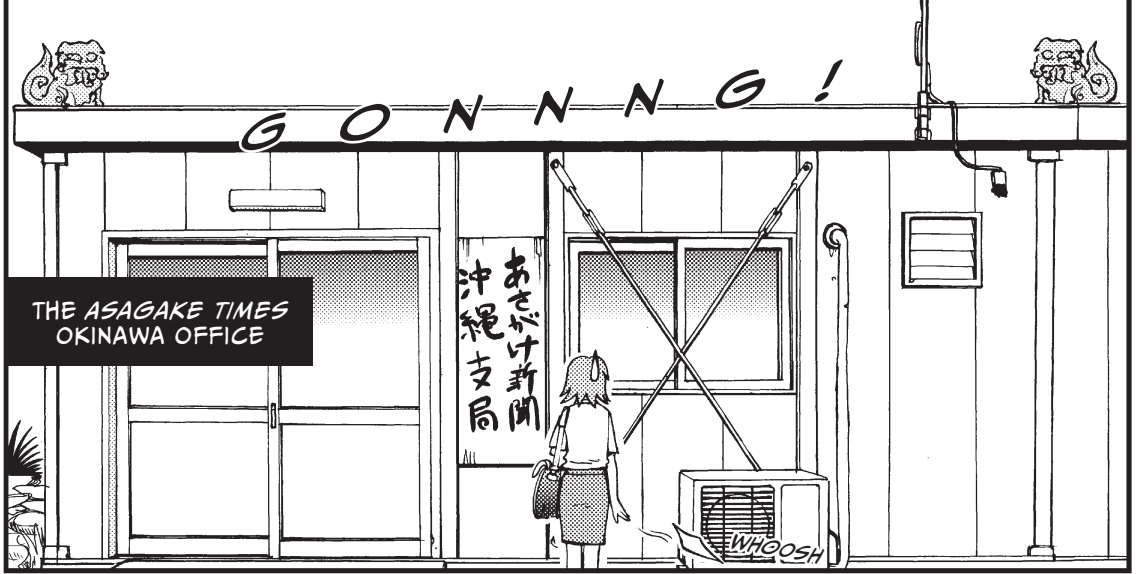


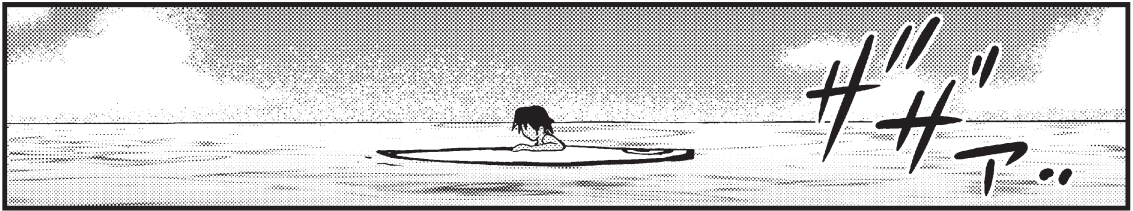
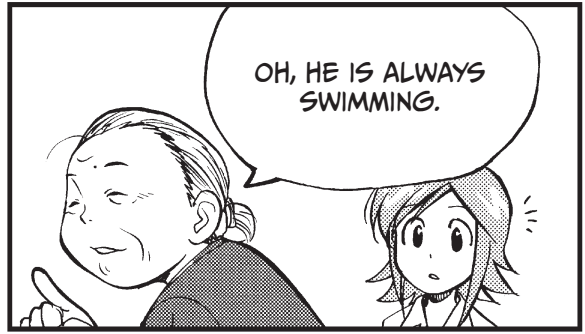
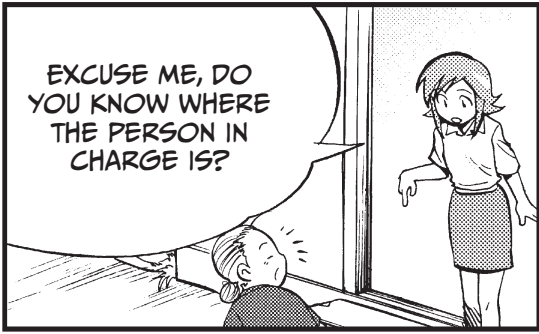
NO MATTER WHERE  
THEY PUT ME, I'LL DO  
MY BEST.

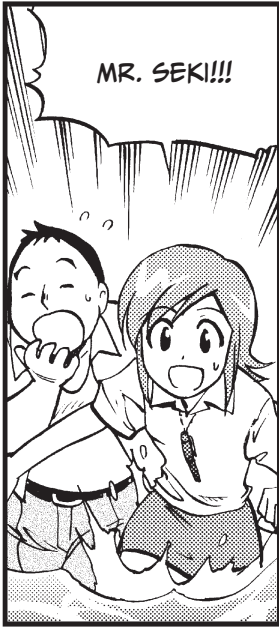


WELL, WHERE IS  
THE ASAGAKE  
TIMES OKINAWA  
OFFICE?

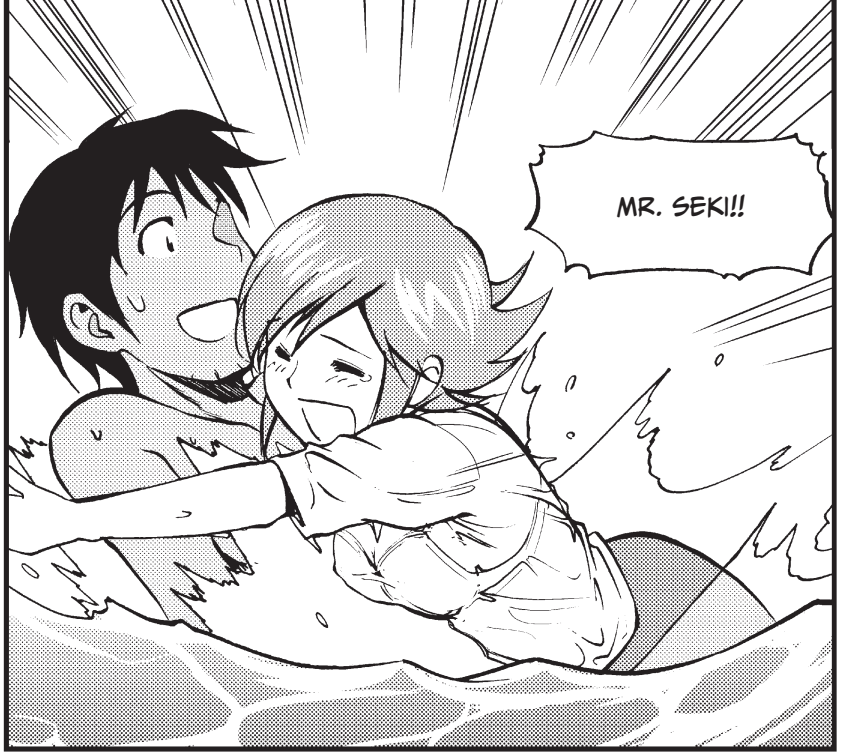




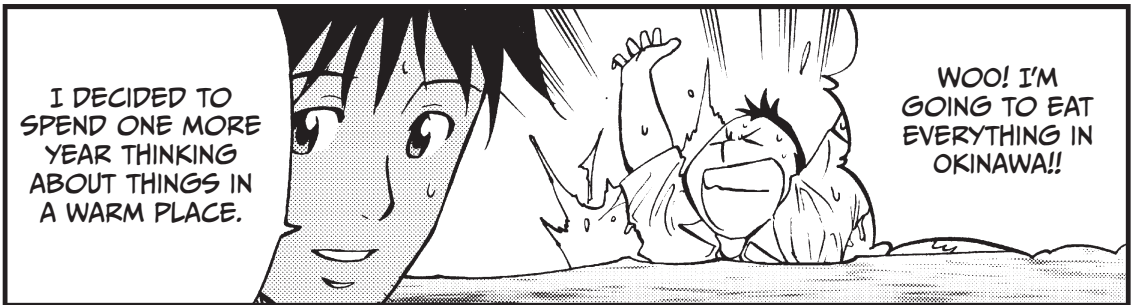




MR. SEKI!!!

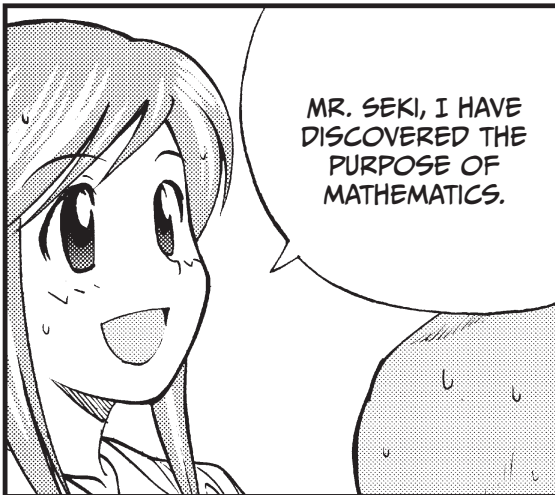


MR. SEKI!!

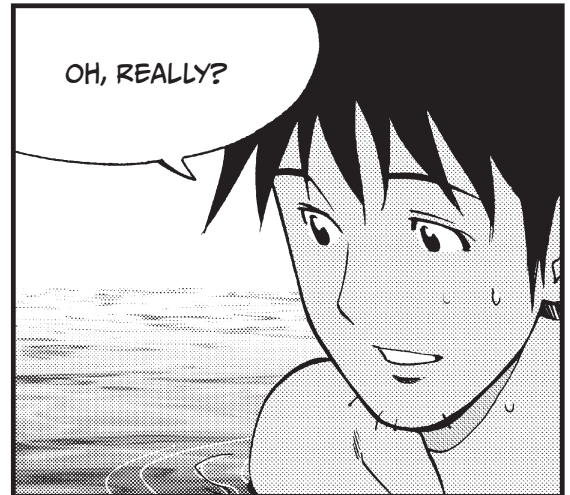


I DECIDED TO SPEND ONE MORE YEAR THINKING ABOUT THINGS IN A WARM PLACE.

WOO! I'M GOING TO EAT EVERYTHING IN OKINAWA!!

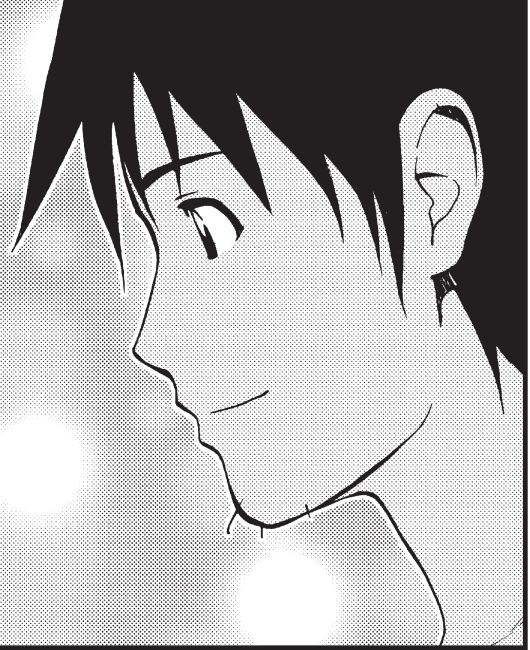


MR. SEKI, I HAVE DISCOVERED THE PURPOSE OF MATHEMATICS.

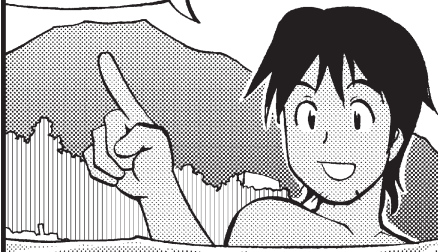


OH, REALLY?

TO CONVEY THINGS  
THAT CANNOT BE  
CONVEYED IN WORDS.



WELL, THEN, NORIKO,  
SUPPOSE THE  
HORIZON IS THE  
X-AXIS...



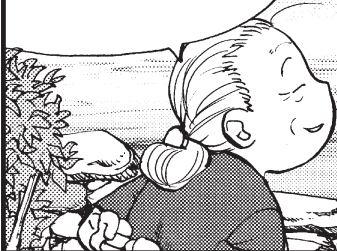
WHAT?



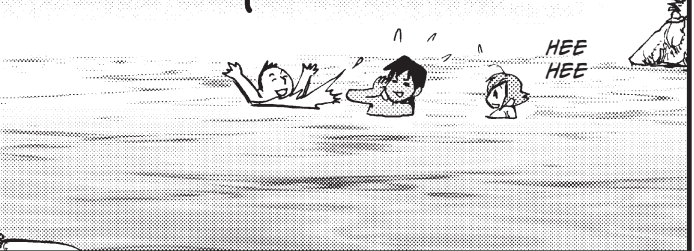
WHAT ARE WE  
GOING TO EAT  
TONIGHT? MMMM,  
NOODLES SOUND  
GOOD.



TOMORROW WILL  
BE ANOTHER  
GREAT DAY.



7-1



# A

## SOLUTIONS TO EXERCISES

### PROLOGUE

1. Substituting

$$y = \frac{5}{9}(x - 32) \text{ in } z = 7y - 30, z = \frac{35}{9}(x - 32) - 30$$

### CHAPTER 1

1. A.  $f(5) = g(5) = 50$   
B.  $f'(5) = 8$

$$\begin{aligned} 2. \quad \lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \frac{(a + \varepsilon)^3 - a^3}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{3a^2\varepsilon + 3a\varepsilon^2 + \varepsilon^3}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} (3a^2 + 3a\varepsilon + \varepsilon^2) = 3a^2 \end{aligned}$$

Thus, the derivative of  $f(x)$  is  $f'(x) = 3x^2$ .

### CHAPTER 2

1. The solution is

$$f'(x) = -\frac{(x^n)'}{(x^n)^2} = -\frac{nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}}$$

$$2. \quad f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$$

When  $x < -2$ ,  $f'(x) > 0$ , when  $-2 < x < 2$ ,  $f'(x) < 0$ , and when  $x > 2$ ,  $f'(x) > 0$ . Thus at  $x = -2$ , we have a maximum with  $f(-2) = 16$ , and at  $x = 2$ , we have a maximum with  $f(2) = -16$ .

3. Since  $f(x) = (1-x)^3$  is a function  $g(h(x))$  combining  $g(x) = x^3$  and  $h(x) = 1-x$ .

$$f'(x) = g'(h(x))h'(x) = 3(1-x)^2(-1) = -3(1-x)^2$$

4. Differentiating  $g(x) = x^2(1-x)^3$  gives

$$\begin{aligned} g'(x) &= (x^2)'(1-x)^3 + x^2((1-x)^3)' \\ &= 2x(1-x)^3 + x^2(-3(1-x)^2) \\ &= x(1-x)^2(2(1-x) - 3x) \\ &= x(1-x)^2(2-5x) \end{aligned}$$

$$g'(x) = 0 \text{ when } x = \frac{2}{5} \text{ or } x = 1, \text{ and } g(1) = 0.$$

$$\text{Thus it has the maximum } g\left(\frac{2}{5}\right) = \frac{108}{3125} \text{ at } x = \frac{2}{5}$$

## CHAPTER 3

1. The solutions are

$$\textcircled{1} \quad \int_1^3 3x^2 dx = x^3 \Big|_1^3 = 3^3 - 1^3 = 26$$

$$\begin{aligned} \textcircled{2} \quad \int_2^4 \frac{x^3 + 1}{x^2} dx &= \int_2^4 \left( x + \frac{1}{x^2} \right) dx = \int_2^4 x dx + \int_2^4 \frac{1}{x^2} dx \\ &= \frac{1}{2}(4^2 - 2^2) - \left( \frac{1}{4} - \frac{2}{4} \right) = \frac{25}{4} \end{aligned}$$

$$\textcircled{3} \quad \int_0^5 x + (1+x^2)^7 dx + \int_0^5 x - (1+x^2)^7 dx = \int_0^5 2x dx = 5^2 - 0^2 = 25$$

2. A. The area between the graph of  $y = f(x) = x^2 - 3x$  and the x-axis equals

$$-\int_0^3 x^2 - 3x dx$$

$$\text{B. } -\int_0^3 x^2 - 3x dx = -\left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Bigg|_0^3 = -\frac{1}{3}(3^3 - 0^3) + \frac{3}{2}(3^2 - 0^2) = \frac{9}{2}$$

## CHAPTER 4

1. The solution is

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}\end{aligned}$$

2. Since

$$\begin{aligned}(\tan x)' &= \frac{1}{\cos^2 x} \\ \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx &= \tan \frac{\pi}{4} - \tan 0 = 1\end{aligned}$$

3. From

$$f'(x) = (x)' e^x + x (e^x)' = e^x + x e^x = (1+x)e^x$$

the minimum is

$$f(-1) = -\frac{1}{e}$$

4. Setting  $f(x) = x^2$  and  $g(x) = \ln x$ , integrate by parts.

$$\int_1^e (x^2)' \ln x dx + \int_1^e x^2 (\ln x)' dx = e^2 \ln e - \ln 1$$

Thus,

$$\int_1^e 2x \ln x dx + \int_1^e x^2 \frac{1}{x} dx = e^2$$

$$\begin{aligned}\int_1^e 2x \ln x dx &= -\int_1^e x dx + e^2 = -\frac{1}{2}(e^2 - 1)^2 + e^2 \\ &= \frac{1}{2}e^2 + \frac{1}{2}\end{aligned}$$

## CHAPTER 5

1. For

$$f(x) = e^{-x}, f'(x) = -e^{-x}, f''(x) = e^{-x}, f'''(x) = -e^{-x}$$

$$f(0) = 1, f'(0) = -1, f''(0) = 1, f'''(0) = -1 \dots$$

$$f(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \dots$$

2. Differentiate

$$f(x) = (\cos x)^{-1}, f'(x) = (\cos x)^{-2} \sin x$$

$$\begin{aligned}f''(x) &= 2(\cos x)^{-3}(\sin x)^2 + (\cos x)^{-2} \cos x \\ &= 2(\cos x)^{-3}(\sin x)^2 + (\cos x)^{-1}\end{aligned}$$

$$\text{from } f(0) = 1, f'(0) = 0, f''(0) = 1$$

3. Proceed in exactly the same way as on page 155 by differentiating  $f(x)$  repeatedly. Since you are centering the expansion around  $x = a$ , plugging in  $a$  will let you work out the  $c_n$ s. You should get  $c_n = 1/n! f^{(n)}(a)$ , as shown in the formula on page 159.

## CHAPTER 6

1. For  $f(x, y) = x^2 + 2xy + 3y^2$ ,  $f_x = 2x + 2y$ , and  $f_y = 2x + 6y$ .
2. The total differential of

$$T = 2\pi \sqrt{\frac{L}{g}} = 2\pi g^{-\frac{1}{2}} L^{\frac{1}{2}}$$

is given by

$$dT = \frac{\partial T}{\partial g} dg + \frac{\partial T}{\partial L} dL = -\pi g^{-\frac{3}{2}} L^{\frac{1}{2}} dg + \pi g^{-\frac{1}{2}} L^{-\frac{1}{2}} dL$$

Thus,

$$\Delta T \approx -\pi g^{-\frac{3}{2}} L^{\frac{1}{2}} \Delta g + \pi g^{-\frac{1}{2}} L^{-\frac{1}{2}} \Delta L$$

Substituting  $\Delta g = -0.02g$ ,  $\Delta L = 0.01L$ , we get

$$\begin{aligned} \Delta T &\approx 0.02\pi g^{-\frac{3}{2}} L^{\frac{1}{2}} g + 0.01\pi g^{-\frac{1}{2}} L^{-\frac{1}{2}} L \\ &= 0.03\pi g^{-\frac{1}{2}} L^{\frac{1}{2}} = 0.03 \frac{T}{2} = 0.015T \end{aligned}$$

So  $T$  increases by 1.5%.

3. If we suppose  $y = h(x)$  is the implicit function of  $f(x, y) = c$ .  
Thus, since the left side is a constant in this region,  $f(x, h(x)) = c$   
near  $x$ .

From the chain rule formula

$$\frac{df}{dx} = 0, \frac{df}{dx} = f_x + f_y h'(x) = 0$$

Therefore

$$h'(x) = -\frac{f_x}{f_y}$$



# B

## MAIN FORMULAS, THEOREMS, AND FUNCTIONS COVERED IN THIS BOOK

### LINEAR EQUATIONS (LINEAR FUNCTIONS)

The equation of a line that has slope  $m$  and passes through a point  $(a, b)$ :

$$y = m(x - a) + b$$

### DIFFERENTIATION

#### DIFFERENTIAL COEFFICIENTS

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

#### DERIVATIVES

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Other notations for derivatives

$$\frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx} f(x)$$

#### CONSTANT MULTIPLES

$$\{\alpha f(x)\}' = \alpha f'(x)$$

#### DERIVATIVES OF NTH-DEGREE FUNCTIONS

$$\{x^n\}' = nx^{n-1}$$

#### SUM RULE OF DIFFERENTIATION

$$\{f(x) + g(x)\}' = f'(x) + g'(x)$$

#### PRODUCT RULE OF DIFFERENTIATION

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$$

#### QUOTIENT RULE OF DIFFERENTIATION

$$\left\{\frac{g(x)}{f(x)}\right\}' = \frac{g'(x)f(x) - g(x)f'(x)}{\{f(x)\}^2}$$

#### DERIVATIVES OF COMPOSITE FUNCTIONS

$$\{g(f(x))\}' = g'(f(x))f'(x)$$

#### DERIVATIVES OF INVERSE FUNCTIONS

When  $y = f(x)$  and  $x = g(y)$

$$g'(y) = \frac{1}{f'(x)}$$

#### EXTREMA

If  $y = f(x)$  has a maximum or a minimum at  $x = a$ ,  $f'(a) = 0$ .

$y = f(x)$  is increasing around  $x = a$ , if  $f'(a) > 0$ .

$y = f(x)$  is decreasing around  $x = a$ , if  $f'(a) < 0$ .

#### THE MEAN VALUE THEOREM

For  $a, b$  ( $a < b$ ), there is a  $c$  with  $a < c < b$ , so that

$$f(b) = f'(c)(b - a) + f(a)$$

## DERIVATIVES OF POPULAR FUNCTIONS

#### TRIGONOMETRIC FUNCTIONS

$$\{\cos \theta\}' = -\sin \theta, \{\sin \theta\}' = \cos \theta$$

## EXPONENTIAL FUNCTIONS

$$\{e^x\}' = e^x$$

## LOGARITHMIC FUNCTIONS

$$\{\log x\}' = \frac{1}{x}$$

# INTEGRALS

## DEFINITE INTEGRALS

When  $F'(x) = f(x)$

$$\int_a^b f(x) dx = F(b) - F(a)$$

## CONNECTION OF INTERVALS OF DEFINITE INTEGRALS

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

## SUM OF DEFINITE INTEGRALS

$$\int_a^b \{f(x) + g(x)\} dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

## CONSTANT MULTIPLES OF DEFINITE INTEGRALS

$$\int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx$$

## SUBSTITUTION OF INTEGRALS

When  $x = g(y)$ ,  $b = g(\beta)$ ,  $a = g(\alpha)$

$$\int_a^b f(x) dx = \int_\alpha^\beta f(g(y)) g'(y) dy$$

## INTEGRATION BY PARTS

$$\int_a^b f'(x) g(x) dx + \int_a^b f(x) g'(x) dx = f(b)g(b) - f(a)g(a)$$

## TAYLOR EXPANSION

When  $f(x)$  has a Taylor expansion near  $x = a$ ,

$$f(x) = f(a) + \frac{1}{1!} f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^{(n)} + \dots$$

TAYLOR EXPANSIONS OF VARIOUS FUNCTIONS

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \dots + (-1)^n \frac{1}{(2n)!} x^{2n} + \dots$$

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots + (-1)^{n-1} \frac{1}{(2n-1)!} x^{2n-1} + \dots$$

$$e^x = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots + \frac{1}{n!} x^n + \dots$$

$$\ln(1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \dots + (-1)^{n+1} \frac{1}{n} x^n + \dots$$

## PARTIAL DERIVATIVES

PARTIAL DERIVATIVES

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

TOTAL DIFFERENTIALS

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

FORMULA OF CHAIN RULE

When  $z = f(x, y)$ ,  $x = a(t)$ ,  $y = b(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{da}{dt} + \frac{\partial f}{\partial y} \frac{db}{dt}$$

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## DIFFERENTIAL AND INTEGRAL CALCULUS DANCE SONG FOR TRIGONOMETRIC FUNCTIONS

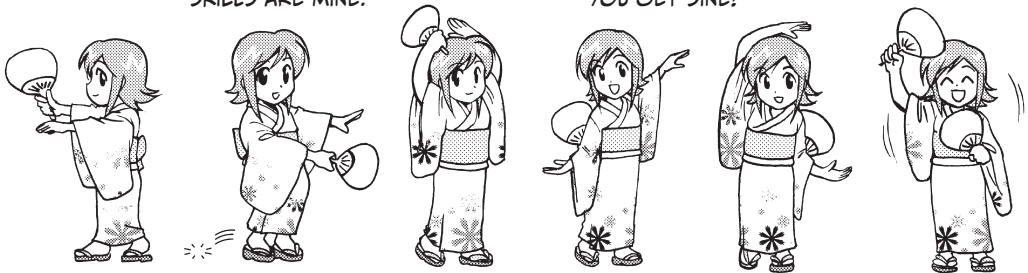
IT SOUNDS QUITE BOOKISH,  
INDEED. BUT THIS DANCE  
SONG MAKES IT SO EASY!

DIFFERENTIATE SINE AND  
YOU GET COSINE!

IN THE LOOP OF SINE AND  
COSINE, IT'S SO NATURAL—  
SOLUTIONS TAKE TURNS  
FOR THE DIFFERENTIAL AND  
INTEGRAL.

IN THE LOOP OF SINE AND  
COSINE, DIFFERENTIAL  
AND INTEGRAL CALCULUS  
SKILLS ARE MINE.

INTEGRATE COSINE AND  
YOU GET SINE!



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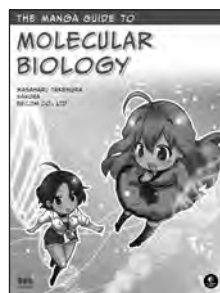
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# A CARTOON GUIDE TO CALCULUS



NORIKO IS JUST GETTING STARTED AS A JUNIOR REPORTER FOR THE *ASAGAKE TIMES*. SHE WANTS TO COVER THE HARD-HITTING ISSUES, LIKE WORLD AFFAIRS AND POLITICS, BUT DOES SHE HAVE THE SMARTS FOR IT? THANKFULLY, HER OVERBEARING AND MATH-MINDED BOSS, MR. SEKI, IS HERE TO TEACH HER HOW TO ANALYZE HER STORIES WITH A MATHEMATICAL EYE.

IN *THE MANGA GUIDE TO CALCULUS*, YOU'LL FOLLOW ALONG WITH NORIKO AS SHE LEARNS THAT CALCULUS IS MORE THAN JUST A CLASS DESIGNED TO WEED OUT WOULD-BE SCIENCE MAJORS. YOU'LL SEE THAT CALCULUS IS A USEFUL WAY TO UNDERSTAND THE PATTERNS IN PHYSICS, ECONOMICS, AND THE WORLD AROUND US, WITH HELP FROM REAL-WORLD EXAMPLES LIKE PROBABILITY, SUPPLY AND DEMAND CURVES, THE ECONOMICS OF POLLUTION, AND THE DENSITY OF SHOCHU (A JAPANESE LIQUOR).

MR. SEKI TEACHES NORIKO HOW TO:

- USE DIFFERENTIATION TO UNDERSTAND A FUNCTION'S RATE OF CHANGE
- APPLY THE FUNDAMENTAL THEOREM OF CALCULUS, AND GRASP THE RELATIONSHIP BETWEEN A FUNCTION'S DERIVATIVE AND ITS INTEGRAL
- INTEGRATE AND DIFFERENTIATE TRIGONOMETRIC AND OTHER COMPLICATED FUNCTIONS
- USE MULTIVARIATE CALCULUS AND PARTIAL DIFFERENTIATION TO DEAL WITH TRICKY FUNCTIONS
- USE TAYLOR EXPANSIONS TO ACCURATELY IMITATE DIFFICULT FUNCTIONS WITH POLYNOMIALS

WHETHER YOU'RE STRUGGLING THROUGH A CALCULUS COURSE FOR THE FIRST TIME OR YOU JUST NEED A PAINLESS REFRESHER, YOU'LL FIND WHAT YOU'RE LOOKING FOR IN *THE MANGA GUIDE TO CALCULUS*.



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